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A Study on CFL Conditions for the DG Solution of Conservation Laws on Adaptive Moving Meshes

Min Zhang[1], Weizhang Huang[2] and Jianxian Qiu[3,*]

[1] *School of Mathematical Sciences, Peking University, Beijing 100871, China*
[2] *Department of Mathematics, University of Kansas, Lawrence, Kansas 66045, USA*
[3] *School of Mathematical Sciences, Fujian Provincial Key Laboratory of Mathematical Modeling, High-Performance Scientific Computing, Xiamen University, Xiamen, Fujian 361005, China*

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Abstract. The selection of time step plays a crucial role in improving stability and efficiency in the Discontinuous Galerkin (DG) solution of hyperbolic conservation laws on adaptive moving meshes that typically employs explicit stepping. A commonly used selection of time step is a direct extension based on Courant-Friedrichs-Levy (CFL) conditions established for fixed and uniform meshes. In this work, we provide a mathematical justification for those time step selection strategies used in practical adaptive DG computations. A stability analysis is presented for a moving mesh DG method for linear scalar conservation laws. Based on the analysis, a new selection strategy of the time step is proposed, which takes into consideration the coupling of the α-function (that is related to the eigenvalues of the Jacobian matrix of the flux and the mesh movement velocity) and the heights of the mesh elements. The analysis also suggests several stable combinations of the choices of the α-function in the numerical scheme and in the time step selection. Numerical results obtained with a moving mesh DG method for Burgers' and Euler equations are presented. For comparison purpose, numerical results obtained with an error-based time step-size selection strategy are also given.

AMS subject classifications: 65M50, 65M60

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1. Introduction

We are concerned with the stability of the discontinuous Galerkin (DG) solution of

*Corresponding author. *Email addresses:* minzhang@math.pku.edu.cn (M. Zhang), jxqiu@xmu.edu.cn (J. Qiu), whuang@ku.edu (W. Huang)

conservation laws on adaptive moving meshes. The DG method is a powerful numerical tool for use in the simulation of hyperbolic problems. It was first used by Reed and Hill [31] for the steady radiation transport equation and studied theoretically by Lesaint and Raviart [25]. The method was extended to conservation laws by Cockburn and Shu (and their coworkers) in a series of papers [6–11]. The DG method has the advantages of high-order accuracy, geometric flexibility, easy use with mesh adaptivity, local data structure, high parallel efficiency, and a good foundation for theoretical analysis. The DG method has been used widely in scientific and engineering computation. Meanwhile, conservation laws typically exhibit discontinuous structures such as shock waves, rarefaction waves, and contact discontinuities and are amenable to mesh adaptation in their numerical solution to enhance numerical resolution and computational efficiency. It is natural to combine the DG method with mesh adaptation method for the solution of conservation laws.

A large amount of work has been done in this area. For example, Bey and Oden [3] combined the hp-method with the DG method for conservation laws and Li and Tang [26] solved two-dimensional conservation laws using a rezoning moving mesh DG method where the physical variables are interpolated from the old mesh to the new one using conservative interpolation schemes. Mackenzie and Nicola [29] solved the Hamilton-Jacobi equation by the DG method using a moving mesh method based on the moving mesh partial differential equation (MMPDE) strategy [21, 22]. Vilar *et al.* [36] studied a DG discretization for solving the two-dimensional gas dynamics equations in Lagrangian formulation. More recently, Uzunca *et al.* [35] employed a moving mesh symmetric interior penalty Galerkin method (SIPG) to solve PDEs with traveling waves. Luo *et al.* considered a quasi-Lagrange moving mesh DG method (MMDG) for conservation laws [27] and multi-component flows [28]. Zhang *et al.* studied the MMDG solution for the radiative transfer equation [38, 39] and shallow water equations (SWEs) [40, 41]. Zhang *et al.* [43] develop a arbitrary Lagrangian-Eulerian discontinuous Galerkin (ALE-DG) methods for the SWEs. Wang *et al.* [37] developed a reconstructed DG Method for compressible flows in Lagrangian formulation.

In principle, any marching scheme (e.g., see Hairer and Wanner [16]) can be used for the time integration of DG computations of hyperbolic conservation laws, including explicit and implicit Runge-Kutta methods [13, 14] and multi-step methods [32]. Nevertheless, explicit schemes have been widely used in these computations. There are at least two considerations for this. First, as we can see later, the stability condition for explicit schemes when applied to hyperbolic equations typically requires the time step-size to be proportional to the minimum mesh element size, which is considered acceptable in practical computations with a uniform mesh. Second, due to the highly nonlinear and hyperbolic nature of conservation laws, there exists hardly any efficient solver for nonlinear algebraic systems (whose linearization is typically non-symmetric and non-definite, and whose solution contains discontinuity, such as shock wave) resulting from the implicit temporal discretization. As such, it does not seem worth the trouble to increase the time step-size using implicit schemes when a uniform mesh is