CSIAM Trans. Appl. Math.
doi: 10.4208/csiam-am.SO-2022-0012

Vol. **4**, No. 2, pp. 225-255
June 2023

Iterative Runge-Kutta-Type Methods with Convex Penalty for Inverse Problems in Hilbert Spaces

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Received 18 March 2022; Accepted 31 August 2022

Abstract. An s-stage Runge-Kutta-type iterative method with the convex penalty for solving nonlinear ill-posed problems is proposed and analyzed in this paper. The approach is developed by using a family of Runge-Kutta-type methods to solve the asymptotical regularization method, which can be seen as an ODE with the initial value. The convergence and regularity of the proposed method are obtained under certain conditions. The reconstruction results of the proposed method for some special cases are studied through numerical experiments on both parameter identification in inverse potential problem and diffuse optical tomography. The numerical results indicate that the developed methods yield stable approximations to true solutions, especially the implicit schemes have obvious advantages on allowing a wider range of step length, reducing the iterative numbers, and saving computation time.

AMS subject classifications: 65J22, 35R30

Key words: Nonlinear ill-posed problem, iterative regularization method, convex penalty, diffuse optical tomography.

1 Introduction

Inverse problems arising from many practical applications can be regarded to solve a nonlinear ill-posed operator equation. This operator equation often builds the map from parameter or source information to boundary data based on the corresponding physical models which are usually partial differential equations (PDEs), such as diffuse optical

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tomography for medical imaging [1], inverse potential problems in geophysics [13] and so on. We consider the ill-posed operator equation

$$F(x)=y, \tag{1.1}$$

where $F:\mathscr{D}(F)\subset X\to Y$ is a nonlinear operator, X and Y are infinite dimensional Hilbert spaces. Instead of exact data y, we are given noisy data y^δ satisfying

$$\|y-y^\delta\|\le\delta \tag{1.2}$$

with known noise level δ. Such inverse problems are considered to be ill-posed, whose solution does not depend continuously on the observation data. Consequently, it is necessary to apply regularization methods [7]. Regularization methods can be seen in three categories: variational regularization methods [27], iterative regularization methods [18] and asymptotical regularization methods [3,20,24,29,36,37], in which asymptotical regularization can be considered as a continuous analogue of iterative methods. For example, recalling the Landweber iteration [12]

$$x_{n+1}^\delta=x_n^\delta-t_n^\delta F'\big(x_n^\delta\big)^*\big(F(x_n^\delta)-y^\delta\big),$$

where t_n^δ is step length. By introducing an artificial time variable, the first-order asymptotical regularization was investigated

$$\frac{dx^\delta(t)}{dt}=-F'\big(x^\delta(t)\big)^*\big(F\big(x^\delta(t)\big)-y^\delta\big),\quad 0<t\le T. \tag{1.3}$$

The stopping time T plays the regularization parameter role, which is often selected via a standard discrepancy principle [29]. Different numerical schemes, e.g. Runge-Kutta (RK)-type methods, can be chosen to solve this initial value problem of ordinary differential equations (ODE) (1.3), sequentially some new type iterative regularization methods are obtained. The theoretical results of the RK regularization methods were well-developed for linear inverse problems [25, 38]. For nonlinear inverse problems, the theories of some special cases of RK methods were studied, such as explicit Euler method (Landweber iteration) [12], linearly implicit Euler method (Levenberg-Marquardt iteration) [11], a variant of the exponential Euler method [14] and particular explicit 2-stage RK Landweber methods [19, 34]. General RK methods were analysed in [4, 22, 23] for nonlinear ill-posed problems.

In this paper, we are interested in reconstructing special features of solutions such as piecewise constancy or sparsity. To do this, the non-smooth penalty terms which include the total variation-like or the L_1-like penalty functionals were introduced.

Let $\Theta:X\to(-\infty,+\infty)$ be a proper, lower semi-continuous, uniformly convex function, and the initial pairs are given as $x_0^\delta=x_0$ and $\xi_0^\delta=\xi_0\in\partial\Theta(x_0)$, the Landweber type method with convex penalty [16] can be presented as

$$\begin{aligned}&\xi_{n+1}^\delta=\xi_n^\delta-t_n^\delta F'\big(x_n^\delta\big)^*\big(F(x_n^\delta)-y^\delta\big),\\&x_{n+1}^\delta=\nabla\Theta^*\big(\xi_{n+1}^\delta\big),\end{aligned} \tag{1.4}$$