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Order Two Superconvergence of the CDG Finite Elements on Triangular and Tetrahedral Meshes

Xiu Ye[1] and Shangyou Zhang[2,*]

[1] *Department of Mathematics, University of Arkansas at Little Rock, Little Rock, AR 72204, USA.*
[2] *Department of Mathematical Sciences, University of Delaware, Newark, DE 19716, USA.*

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Abstract. It is known that discontinuous finite element methods use more unknown variables but have the same convergence rate comparing to their continuous counterpart. In this paper, a novel conforming discontinuous Galerkin (CDG) finite element method is introduced for Poisson equation using discontinuous P_k elements on triangular and tetrahedral meshes. Our new CDG method maximizes the potential of discontinuous P_k element in order to improve the convergence rate. Superconvergence of order two for the CDG finite element solution is proved in an energy norm and in the L^2 norm. A local post-process is defined which lifts a P_k CDG solution to a discontinuous P_{k+2} solution. It is proved that the lifted P_{k+2} solution converges at the optimal order. The numerical tests confirm the theoretic findings. Numerical comparison is provided in 2D and 3D, showing the P_k CDG finite element is as good as the P_{k+2} continuous Galerkin finite element.

AMS subject classifications: 65N15, 65N30

Key words: Finite element, conforming discontinuous Galerkin method, stabilizer free, triangular grid, tetrahedral grid.

1 Introduction

Finite element methods use two different methodologies to approximate the solution of the following second order elliptic problem:

$$-\Delta u = f \quad \text{in } \Omega, \tag{1.1}$$

$$u = g \quad \text{on } \partial\Omega, \tag{1.2}$$

*Corresponding author. *Email addresses:* `xxye@ualr.edu` (X. Ye), `szhang@udel.edu` (S. Zhang)

where Ω is a 2D polygonal domain, or a 3D polyhedral domain. One is the classic conforming finite element method using continuous P_k polynomials for the solution of the PDE (1.1) and another uses discontinuous P_k polynomials. Interior penalty discontinuous Galerkin (IPDG) method in [3] and CDG methods in [28] are the later cases. Since discontinuous P_k polynomial introduces many more degrees of freedom, one would expect higher order convergence for the discontinuous finite element methods. But the current IPDG and CDG methods, using discontinuous P_k elements, have their convergence rates the same as their continuous counterpart, i.e. $\mathcal{O}(h^k)$ in the energy norm and $\mathcal{O}(h^{k+1})$ in the L^2 norm.

In this paper, we will introduce a new CDG method that uses discontinuous P_k polynomials to approximate the solution of the problem (1.1)-(1.2) on triangular and tetrahedral meshes. The finite element formulations of discontinuous element methods tend to be more complex to ensure weak continuity of discontinuous solutions. Sometimes penalty parameter is needed and to be tuned. However our new CDG method has the following simple formulation without stabilizers and penalty parameters: Find $u_h \in V_h$

$$(\nabla_w u_h, \nabla_w v) = (f, v), \quad \forall v \in V_h, \tag{1.3}$$

where ∇_w is the so called weak gradient. In addition, this new CDG method fully utilizes the degrees of freedom of discontinuous P_k polynomial to obtain order two superconvergence of $\mathcal{O}(h^{k+2})$ in an energy norm and $\mathcal{O}(h^{k+3})$ in the L^2 norm. We attribute these good features to the definition of weak derivatives which are specially designed for discontinuous P_k elements. Order two superconvergence of the CDG solution is confirmed theoretically and numerically. Additionally, we define and prove a local post-process which lifts a C^{-1}-P_k CDG solution to a C^{-1}-P_{k+1} with the optimal order convergence of the latter. We may consider this order two superconvergence from the point of the number of degrees of freedom. On triangular meshes, the numbers of degree of freedom of the space C^{-1}-P_k and the space C^0-P_{k+2} are

$$Ch^{-2}(k+1)(k+2), \quad Ch^{-2}(k+2)^2,$$

respectively. On tetrahedral meshes, they are

$$Ch^{-3}(k+2)(k^2+4k+3), \quad Ch^{-3}(k+2)(k^2+4k+4),$$

respectively. In both cases the dimension of the P_k discontinuous polynomial space is slightly less than that of the P_{k+2} continuous polynomial space. But we show both theoretically and numerically that the CDG P_k solution converges as well as the continuous Galerkin P_{k+2} solution.

The weak derivative used in CDG methods was first introduced in the weak Galerkin (WG) finite element method [24]. Then WG methods have been applied for solving various PDEs such as Sobolev equation, the Navier-Stokes equations, the Oseen equations, time-dependent Maxwell's equations, elliptic interface problems, biharmonic equations, etc, [4–6, 8–27, 31, 32]. The CDG method is derived from the WG method by eliminating