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Boundedness of Some Commutators of Marcinkiewicz Integrals on Hardy Spaces

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Abstract. Based on the results of the boundedness of μ_Ω^b on L^p spaces, by using the theory of atomic decomposition of Hardy spaces, we obtain the boundedness of μ_Ω^b on Hardy spaces.

Key Words: Marcinkiewicz integral, commutator, Lipschitz space, Hardy space.

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1 Introduction

Suppose that S^{n-1} is the unit sphere of $\mathbf{R^n}(n\geq 2)$ equipped with normalized Lebesgue measure. Let $\Omega\in L^1(S^{n-1})$ be homogeneous function of degree zero and

$$\int_{S^{n-1}}\Omega(x')d\sigma(x')=0, \tag{1.1}$$

where $x'=x/|x|$ for any $x\neq 0$.

The Marcinkiewicz integral is defined by

$$\mu_\Omega(f)(x)=\Big(\int_0^\infty |F_{\Omega,t}(f)(x)|^2\frac{dt}{t^3}\Big)^{1/2},$$

where

$$F_{\Omega,t}(f)(x)=\int_{|x-y|\leq t}\frac{\Omega(x-y)}{|x-y|^{n-1}}f(y)dy.$$

Let $b\in \mathrm{L}^1_{loc}(\mathbf{R^n})$, the commutator generated by the Marcinkiewicz integral μ_Ω and b is defined by

$$\mu_{\Omega,b}(f)(x)=\Big(\int_0^\infty |F_{\Omega,b,t}(f)(x)|^2\frac{dt}{t^3}\Big)^{1/2},$$

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where

$$F_{\Omega,b,t}(f)(x)=\int_{|x-y|\le t}\frac{\Omega(x-y)}{|x-y|^{n-1}}(b(x)-b(y))f(y)dy.$$

Y. Ding [1] studied the continuity properties of higher order commutators generated by the homogeneous fractional integral and BMO functions on certain Hardy spaces, the special case of the main result in [1] is the following theorem:

Theorem 1.1 ([1]). *Let $b\in BMO(\mathbf{R^n})$, $0<\mu<n$ and $\Omega\in L^r(S^{n-1})(r>n/(n-\mu))$. If $\omega_r(\delta)$ satisfy*

$$\int_0^1\frac{\omega_r(\delta)}{\delta}\Big(\log\frac{1}{\delta}\Big)^m d\delta<\infty, \tag{1.2}$$

then $T^{b,m}_{\Omega,\mu}$ is bounded from $H^1_{b^m}(\mathbf{R^n})$ to $L^{n/(n-\mu)}(\mathbf{R^n})$, where

$$T^{b,m}_{\Omega,\mu}(f)(x)=\int_{\mathbf{R^n}}\frac{\Omega(x-y)}{|x-y|^{n-\mu}}(b(x)-b(y))^m f(y)dy,\quad m\in\mathbf{N}.$$

In 2007, H. Wang [2] gave the $(H^1,L^{n/(n-\beta)})$ type estimates for $\mu_{\Omega,b}$ with the kernel Ω satisfying the logarithmic type Lipschitz conditions.

Theorem 1.2 ([2]). *Let $b\in Lip_\beta(\mathbf{R^n})$, $0<\beta<1$. If Ω is a homogeneous function of degree zero and satisfies the following conditions:*

(1) $\int_{S^{n-1}}\Omega(x')d\sigma(x')=0$ and $\Omega\in L^r(S^{n-1})$ for some $r\ge n/(n-\beta)$;

(2) there exist constants $C>0$ and $\rho>1$ such that

$$|\Omega(y_1)-\Omega(y_2)|\le\frac{C}{(\ln\frac{2}{|y_1-y_2|})^\rho}$$

for any $y_1,y_2\in S^{n-1}$. Then $\mu_{\Omega,b}$ is bounded from $H^1(\mathbf{R^n})$ to $L^{n/n-\beta}(\mathbf{R^n})$.

In 2011, Y. He [3] obtained the $(L^p(\alpha),L^p(\beta))$ type estimates for $\mu_{\Omega,b}$ with the kernel satisfying the logarithmic type Lipschitz conditions. In 2012, by using Theorem 1.2, Jiang [4] proved that $\mu_{\Omega,b}$ is bounded from $H^1_b(\omega)$ to $L^1(\mathbf{R^n})$.

Theorem 1.3 ([3]). *Let $\Omega\in L^\infty(S^{n-1})$ satisfy the cancellation property* (1.1). *In addition, suppose that there exist constants $C>0$ and $\rho>2$ such that*

$$|\Omega(y_1)-\Omega(y_2)|\le\frac{C}{(\ln\frac{1}{|y_1-y_2|})^\rho} \tag{1.3}$$

hold uniformly in $y_1,y_2\in S^{n-1}$, $1<p<\infty$, $\alpha,\beta\in A_p$, $b\in BMO(\omega)$, $\omega=(\alpha\beta^{-1})^{1/p}$. Then the following inequality hold:

$$||\mu_{\Omega,b}(f)||_{L^p(\beta)}\le C||b||_{BMO(\omega)}||f||_{L^p(\alpha)}.$$