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# THE DISCRETE RAVIART-THOMAS MIXED FINITE ELEMENT METHOD FOR THE $p$-LAPLACE EQUATION

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**Abstract.** We consider the discrete Raviart-Thomas mixed finite element method (dRT-MFEM) for the $p$-Laplace equation in the new sense of measurement. The new measurement of $p$-Laplace equation for $2 \leq p < \infty$ was studied by D. J. Liu (APPL. NUMER. MATH., 152: 323-337, 2020), where the reliable error analysis for conforming and nonconforming FEM were obtained. This paper provide the reliable and efficient error analysis of dRT-MFEM for $p$-Laplace equation ($1 < p < 2$). The numerical investigation for benchmark problem demonstrates the accuracy and robustness of the proposed dRT-MFEM.

**Key words.** Adaptive finite element methods, discrete Raviart-Thomas mixed finite element method, $p$-Laplace equation.

## 1. Introduction

We discuss the following nonlinear $p$-Laplace equation ($1 < p < 2$) in the bounded Lipschitz domain $\Omega \subset \mathbb{R}^2$ with the given $f \in L^q(\Omega)$ ($q$ conjugate of $p$),

$$(1) \qquad \begin{cases} -\mathrm{div}(|\nabla u|^{p-2}\nabla u) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

The $p$-Laplace equation (1) admits a unique weak solution [4] satisfying

$$(2) \qquad u = \arg\min E(v) \quad \text{for } v \in W_0^{1,p}(\Omega) := \{v \in W^{1,p}(\Omega) : v|_{\partial\Omega} = 0\}.$$

where

$$(3) \qquad E(v) := \int_\Omega W(\nabla v)dx - \widetilde{F}(v).$$

The energy density function $W : \mathbb{R}^2 \to \mathbb{R}$ reads $W(a) := |a|^p/p$ with the derivative $\sigma(a) := DW(a) = |a|^{p-2}a$ for all $a \in \mathbb{R}^2 \setminus \{0\}$ which is recorded as $\sigma$ for the convenience of subsequent discussion and $\widetilde{F}(v) := \int_\Omega fv\,dx$ and the dual function

$$W^*(a) := \frac{|a|^q}{q} \qquad \left(\frac{1}{p} + \frac{1}{q} = 1\right).$$

The Euler-Lagrange equation of (2) consists in finding $u \in W_0^{1,p}(\Omega)$ with

$$(4) \qquad \int_\Omega \sigma \cdot \nabla v dx - \widetilde{F}(v) = 0 \quad \text{for all } v \in W_0^{1,p}(\Omega).$$

The finite element analysis for (1) has been well done. We can find some previous work in sense of traditional $W^{1,p}(\Omega)$-norm in [12, 15, 23, 13]. Sharper error estimates were derived in [18, 14, 3] by developing the so called quasi-norm techniques, and these techniques were extended to establish improved a posteriori error estimators of residual type for the adaptive finite element methods [11, 19]. Liu [17, 16] generalized the quasi-norm techniques to a new measure framework for $2 \leq p < \infty$,

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and obtained the reliable error analysis for conforming FEM, nonconforming FEM, and dRT-MFEM. Nevertheless, the research for $1 < p < 2$, which including a singular operator, was not covered in the above references. In this paper, we mainly focus on the dRT-MFEM of $p$-Laplace equation for $1 < p < 2$.

Marini representation [2, 21] was proposed for the purpose of the cost-free approximation of Raviart-Thomas MFEM for linear problem. Arbogast [1] improved the method for general variable coefficients elliptic PDEs. A one-point quadrature rule in the dual Raviart-Thomas MFEM leads to the dRT-MFEM in [10], which developed the Marini representation for nonlinear optimal design problem, the first guaranteed energy bound and an optimal a posteriori error estimate were obtained. Liu [17] generalized the dRT-MFEM for $p$-Laplace equation ($2 \le p < \infty$), and provided the reliable error analysis. This paper will study the dRT-MFEM of $p$-Laplace problem for $1 < p < 2$, and show the error estimators without a gap between the upper bound and the lower bound.

The remaining parts of this paper are organized as follows. Section 2 introduces the newly defined measure to quantify the quality of approximations, and proves the convex control of energy density function $W$. Section 3 states the dRT-MFEM for the $p$-Laplace problem. A priori and a posteriori error estimators based on the newly defined measure are presented in Section 4. Some numerical experiments conclude the paper in Section 5 with empirical evidence of the expected convergence.

Standard notation applies throughout this paper to Lebesgue and Sobolev spaces $L^q(\Omega)$, $H^s(\Omega)$, and $H(\mathrm{div},\Omega)$, as well as to the associated norms $\|\cdot\|_{q,\Omega} := \|\cdot\|_{L^q(\Omega)}$, $|||\cdot|||_{q,\Omega} := \|\nabla\cdot\|_{L^q(\Omega)}$, and $|||\cdot|||_{NC,q,\Omega} := \|\nabla_{NC}\cdot\|_{L^q(\Omega)}$ with the piecewise gradient $\nabla_{NC}\cdot|_T := \nabla(\cdot|_T)$ for all $T$ in a regular triangulation $\mathcal{T}$ of the polygonal domain $\Omega$. Here and throughout,the expression "$\lesssim$" abbreviates an inequality up to some multiplicative generic constant, i.e., $A \lesssim B$ means $A \le C\,B$ with some generic constant $0 \le C < \infty$, which depends on the interior angles of the triangles but not their sizes.

## 2. The convexity control of W

We firstly recall the concept of distance. Define

$$F(a) := |a|^{p/2-1}a \quad \forall a \in L^2(\Omega;\mathbb{R}^2).$$

Let $\alpha := DW(a)$, $\beta := DW(b)$ for $a, b \in L^2(\Omega;\mathbb{R}^2)$, the distance of $F(a)$ and $F(b)$ can be defined as follows [16]

$$\|F(a) - F(b)\|^2_{2,q,\Omega} := \int_\Omega \frac{|\alpha-\beta|^2}{(|\alpha|+|\beta|)^{2-q}}dx \quad \forall a, b \in \mathbb{R}^2. \tag{5}$$

The remaining parts of this section are devoted to the convexity control of energy density function $W$, which is formulated in the following lemma 2.2.

**Lemma 2.1.** *Given $1 < p < 2$ and the conjugate $q$, there exist positive constants $s_1(p)$, $s_2(p)$, $m_1(p)$, $m_2(p)$, $l_1(p)$, $l_2(p)$ such that for any $a, b \in L^2(\Omega;\mathbb{R}^2)$, $\alpha := DW(a)$, $\beta := DW(b)$ satisfy*

$$\begin{aligned} s_1(p)\,(DW(b) - DW(a))\cdot(b-a) &\le |DW(b) - DW(a)|^2\,(|\alpha|+|\beta|)^{q-2} \\ &\le s_2(p)\,(DW(b) - DW(a))\cdot(b-a)\,. \end{aligned} \tag{6}$$

$$\begin{aligned} m_1(p)\,(|b|+|a|)^{p-2}\,|b-a|^2 &\le (DW(b) - DW(a))\cdot(b-a) \\ &\le m_2(p)\,(|b|+|a|)^{p-2}\,|b-a|^2. \end{aligned} \tag{7}$$