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NUMERICAL APPROXIMATIONS FOR THE NULL CONTROLLERS OF STRUCTURALLY DAMPED PLATE DYNAMICS

PELIN G. GEREDELI, CARSON GIVENS AND AHMED ZYTOON*

Abstract. In this paper, we consider a structurally damped elastic equation under hinged boundary conditions. Fully-discrete numerical approximation schemes are generated for the null controllability of these parabolic-like PDEs. We mainly use finite element method (FEM) and finite difference method (FDM) approximations to show that the null controllers being approximated via FEM and FDM exhibit exactly the same asymptotics of the associated minimal energy function. For this, we appeal to the theory originally given by R. Triggiani [20] for construction of null controllers of ODE systems. These null controllers are also amenable to our numerical implementation in which we discuss the aspects of FEM and FDM numerical approximations and compare both methodologies. We justify our theoretical results with the numerical experiments given for both approximation schemes.

Key words. Null control, finite element method, finite difference method.

1. Introduction

The partial differential equations (PDEs) of plate dynamics ubiquitously arise in elasticity to model and describe the oscillations of thin structures with large transverse displacements [10]. Moreover, researchers of PDE control theory are often interested in devising control input methodologies by which one can elicit some pre-assigned behavior with respect to solutions of a given controlled plate or bean PDE system. In the course of constructing such a control theory for the given damped or undamped plate PDE, its underlying characteristics -hyperbolic or parabolic- must necessarily be taken into account [13].

For example, whereas in hyperbolic equations, we have the notion of finite speed of propagation and evolution of singularities, the parabolic equations posses infinite speed of propagation and smoothing effect. In consequence, the notion of exact controllability-i.e., steering initial data to any finite energy state at some time (large enough) - is a reasonable object of study for hyperbolic problems. On the other hand, the null controllability problem- steering the initial data to the zero state at any time- makes sense for parabolic problems due to their smoothing effects.

In particular, there has been a great interest in studying the null controllability of infinite dimensional systems [1, 2, 3, 7, 11, 20] with a view towards attaining optimal estimates for norms of minimal norm steering controls. In particular, null-controllability for deterministic parabolic-like PDE dynamics plays a crucial role in connection with corresponding stochastic parabolic differential equations. For example, it is known that the notion of null-controllability is equivalent to the strong Feller property of the semigroup of transition of the corresponding stochastic differential equation, which is obtained from the deterministic one by simply replacing the deterministic control with stochastic noise [5, 6, 8].

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*Corresponding author.

This manuscript considers certain PDE dynamics which exhibit analytic, or parabolic-like features. Since these dynamics are associated with an infinite speed of propagation (see [12]), it seems natural to ask: "Is there any control function which steers the solution to the zero state after some certain time $T > 0$?" This is the problem of null controllability. However, we must distinguish the "null controllability" concept between finite and infinite dimensional (PDE) systems since while the issue of finding asymptotics for the associated minimal energy function defined in (5) has completely been characterized in the finite dimensional ODE case [17, 18], the infinite dimensional PDE case is in general an open problem. [17] provides a formula which describes the growth of the minimal norm control, as time $T \to 0$ for ODE dynamics. This result depends on the Kalmans rank condition, which is the sufficient and necessary controllability condition in finite dimensions. In the case of interior boundary control, it was proved in [20] there is a relation between the infinite dimensional asymptotics and finite dimensional truncations such that a priori bounds manifested by the approximating sequence of null controllers (for finite dimensional system) will lead to the conclusion of a null controller for the (infinite dimensional) analytic PDE systems under consideration. It was also shown in [20] that infinite dimensional null controllers will capture the sharp asymptotics of the associated minimal energy function, which is defined through the means of minimal norm controls (see (5)).

The numerical approximation of controlled PDEs has been a topic of longstanding interest [7] however in contrast to the growing literature on theoretical results obtained for the null controllability of parabolic-like plate equations, the knowledge about numerical approximation of the null controllability of PDE dynamics which exhibit analytic, or parabolic-like features is relatively limited. In [1] semidiscrete finite element method (FEM) approximation scheme were presented for the null controllability of non-standard parabolic PDE systems. The key feature in [1] is that the approximating null controllers exhibit the asymptotics of the associated minimal energy function for the fully infinite dimensional system.

In this manuscript, our main goals are to derive fully-discrete Finite Element Method (FEM) and Finite Difference Method (FDM) numerical approximation schemes for a certain (nonstandard) analytic and parabolic-like PDE system, give numerical implementation, and compare the respective FEM and FDM approximations for this controlled structurally damped elastic equation. The main novelties of the current work are:

(i) Fully discrete FEM Approximation: The PDE model given in (1) below was firstly studied in [1]. It was proved that certain finite element method (FEM) approximations $\{u_N^*\}$ and their limiting controller $\{u^*\}$ for the structurally damped PDE (1) manifest the asymptotics (given in Theorem 1.1) of $\mathcal{E}_{min}(T)$ defined in (5). However, in this work no numerical implementation was provided for the derived FEM scheme. In the present work, unlike the semi-discrete approximations, we use "fully-discrete" FEM approximation and provide a numerical experiment to justify that the approximation of the null controllers, within FEM numerical scheme framework, obey the same blow up rate of $\mathcal{O}(T^{-3/2})$ given in Theorem 1.1. Moreover, we give an explicit formula for the approximate control functions.

(ii) Fully discrete FDM Approximation: We numerically analyze the null controllability problem for the given PDE (1) below by means of the finite difference method approximation scheme. We see that Theorem 1.2 can be employed