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Advance in Operator Theory and Jiang's Seven Questions

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In memory of Professor Jiang Zejian. Dedicated to the 70th Anniversary of the Founding of Mathematics Discipline in Jilin University.

Abstract. In this paper we review the seven questions posed by Professor Jiang Zejian and summarize the progress of them. Moreover, we introduce those applications of tools developed in studying Jiang's questions, especially in dealing with the Halmos' third problem.

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1 Jiang's seven questions

From the late 1970's to the early 1990', a number of questions posed by Jiang is folklore on the topic of functional analysis. In particular, Jiang thought deeply about the structure of operators on a Hilbert space $\mathcal{H}$. What should be the basic elements of the structure of operators on a separable infinite dimensional Hilbert

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space $\mathcal{H}$? And can we use these basic elements to establish a representation theory of operators? This is not only a mathematical question but also a philosophical one.

What is the basic elements in the similarity classification theory of matrices? If we ask this question, then the natural answer must be the "Jordan blocks". The well-known Jordan canonical form theorem tells us that an $n\times n$ complex matrix M is always similar to a direct sum of some Jordan blocks. Each Jordan block has the following features:

(1) It is related to its eigenvalues.

(2) Its size is depended on the dimension of its eigen-space.

(3) It can not be written into a direct sum of smaller blocks.

Denote by $J_k(\lambda)$ the Jordan matrix with eigenvalue λ acting on the k-dimensional eigen-subspace $\mathbb{C}^k$, we define the commutant algebra of $J_k(\lambda)$ as

$$\mathcal{A}'\big(J_k(\lambda)\big)=\big\{A\in M_{k\times k}(\mathbb{C});AJ_k(\lambda)=J_k(\lambda)A\big\}.$$

The only idempotents in $\mathcal{A}^{'}(J_k(\lambda))$ are 0 and the identity I_k. Conversely, if the commutant of a $k\times k$ complex matrix T contains only two idempotents 0 and I, then T must be similar to the Jordan block $J_k(\lambda)$ and λ is its unique eigenvalue. From this point of view, Jiang proposed the concept of BIR operator. Denote the set of all bounded linear operators on a separable Hilbert space $\mathcal{H}$ by $B(\mathcal{H})$.

Definition 1.1. *An operator $T\in B(\mathcal{H})$ is called a BIR operator if and only if its commutant $\mathcal{A}^{'}(T)$ contains only two trivial idempotents* 0 *and the identity operator I.*

Jiang pointed out that there are some normal operators which cannot be written as a direct sum (finite or infinite) of BIR operators. This fact depends on the structure of operators on the infinite dimensional Hilbert space. For example, there are lots of operators having no eigenvalue, that is, $\sigma_p(T)=\emptyset$.

At around the same time, Gilfeather, a British mathematician, gave a similar definition, called strongly irreducible operator [7].

Definition 1.2. *An operator $T\in B(\mathcal{H})$ is called strongly irreducible, if XTX^{-1} is irreducible for any invertible operator $X\in B(\mathcal{H})$.*

It is equivalent to say that there are no nontrivial orthogonal projections (in other words, except 0 and identity operator I) in the commutant $\mathcal{A}^{'}(XTX^{-1})$. One can easily verify that above two definitions are equivalent. However, Jiang