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Linearized Transformed $L1$ Galerkin FEMs with Unconditional Convergence for Nonlinear Time Fractional Schrödinger Equations

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Abstract. A linearized transformed $L1$ Galerkin finite element method (FEM) is presented for numerically solving the multi-dimensional time fractional Schrödinger equations. Unconditionally optimal error estimates of the fully-discrete scheme are proved. Such error estimates are obtained by combining a new discrete fractional Grönwall inequality, the corresponding Sobolev embedding theorems and some inverse inequalities. While the previous unconditional convergence results are usually obtained by using the temporal-spatial error spitting approaches. Numerical examples are presented to confirm the theoretical results.

AMS subject classifications: 34A08, 65M12, 65M60, 65N30

Key words: Optimal error estimates, time fractional Schrödinger equations, transformed $L1$ scheme, discrete fractional Grönwall inequality.

1. Introduction

In this paper, we propose a linearized scheme for numerically solving the following nonlinear time fractional Schrödinger equations (TFSEs):

$$\mathbf{i}\partial_t^\alpha u + \Delta u + |u|^2 u = 0, \quad (x,t) \in \Omega \times (0,T], \tag{1.1}$$

$$u(x,0) = u_0(x), \quad x \in \Omega, \tag{1.2}$$

$$u(x,t) = 0, \quad x \in \partial\Omega, \quad t \in [0,T], \tag{1.3}$$

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where $\mathbf{i} = \sqrt{-1}$, $\Omega \in \mathbb{R}^d\,(d = 2,3)$ is a bounded convex and smooth polygon/polyhedron and $u(x,t) \in \Omega \times [0,T]$ is a complex function. The Caputo fractional derivative ∂_t^α is denoted as

$$\partial_t^\alpha u(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x,z)}{\partial z} \frac{1}{(t-z)^\alpha} dz, \quad \alpha \in (0,1),$$

where $\Gamma(\cdot)$ denotes the Gamma function. The TFSEs can be obtained by using a fractional variational principle [25]. The TFSEs were first introduced in [26] to describe the non-Markovian evolution of a free particle. Then Iomin [9] discussed the physical relevance of the TFSEs in quantum mechanics and found that the equations were a particular case of the quantum comb model. Tofighi [34] considered the motion of a particle under the influence of a real potential. More references and related applications can be seen in [18,20,35,37].

Recently, many numerical schemes have been constructed for solving the TFSEs. In [37], Wei *et al.* developed an implicit fully discrete local discontinuous Galerkin (LDG) FEM for solving the one-dimensional TFSEs. They applied the $L1$ finite difference scheme in time and LDG FEMs in space. The error estimate is given by considering the linear problems. In [24], Mohebbi *et al.* proposed an efficient numerical scheme by applying the meshless method in space and $L1$ difference scheme in time for the TFSEs. In [2], Bhrawy *et al.* used a shifted Legendre collocation method in two consecutive steps to numerically solve one and two dimensional TFSEs when considering the initial-boundary and non-local conditions. In [4], Esen *et al.* applied the quadratic B-spline Galerkin method to solve the TFSEs. In [23], Liu *et al.* developed an efficient numerical scheme by reproducing kernel theory in time and using collocation method in the spatial direction. More details on numerical methods for the models can be found in [5,6,36,38,39]. In these references, the convergence results were either missing or obtained under certain spatial-temporal stepsize restrictions.

In order to remove the spatial-temporal stepsize restrictions, the temporal-spatial error splitting argument was proposed to get the unconditionally convergent results for inter-order PDEs in [12, 13]. Besides, some approaches by using error estimates in certain norms and Sobolev embedding theorem were applied, see e.g., [7, 14, 32]. Recently, the temporal-spatial error splitting argument was widely used for analyzing the time fractional problems. The earlier work can be found in [8, 16, 19], where the unconditional convergence results were obtained by assuming that the solutions are smooth. Taking the initial singularity into account, some researchers obtained the unconditionally convergent results of the $L1$ scheme [17,29] and the Alikhanov scheme [41] based on graded meshes, i.e.

$$t_n = T\left(\frac{n}{N}\right)^\gamma, \quad n = 1,2,\ldots,N.$$

The optimal convergence order of the L1 scheme [21, 31] and the Alikhanov scheme [1,22,28] can be $2-\alpha$ and 2 if

$$\gamma \geq \frac{2-\alpha}{\alpha} \quad \text{and} \quad \gamma \geq \frac{2}{\alpha}.$$