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An Algebraic Multigrid-Based Physical Factorization Preconditioner for the Multi-Group Radiation Diffusion Equations in Three Dimensions

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Abstract. The paper investigates the robustness and parallel scaling properties of a novel physical factorization preconditioner with algebraic multigrid subsolves in the iterative solution of a cell-centered finite volume discretization of the three-dimensional multi-group radiation diffusion equations. The key idea is to take advantage of a particular kind of block factorization of the resulting system matrix and approximate the left-hand block matrix selectively spurred by parallel processing considerations. The spectral property of the preconditioned matrix is then analyzed. The practical strategy is considered sequentially and in parallel. Finally, numerical results illustrate the numerical robustness, computational efficiency and parallel strong and weak scalabilities over the real-world structured and unstructured coupled problems, showing its competitiveness with many existing block preconditioners.

AMS subject classifications: 65F08, 65N08, 65N55, 65Y05

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1. Introduction

The radiative transport model plays a momentous role in massive star formation, optical remote sensing, inertial confinement fusion and high energy density physics.

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Targeted at simulating the computationally expensive frequency-dependent radiative transfer in three dimensions, particular effort has been made in the past several decades to develop mathematically less complicated yet numerically accurate approximations. One of the simplest and most extensively used treatments is the flux-limited multi-group diffusion approximation [24], where the associated radiation quantities are classified into a finite number of energy groups and averaged over the frequency bands. The governing equations are a cluster of second-order highly nonlinear and discontinuous parabolic partial differential equations (PDEs) with sophisticated and time-varying couplings. This PDE system is often referred to as the multi-group radiation diffusion (MGD) equations in a domain with many materials, whose solution exhibits multiple spatio-temporal scales and wave-like propagation characteristics [8]. Hence, it is a challenging task, despite of the enormous development around this issue in recent years, to search for the robust, accurate and reliable solutions of the associated initial boundary value problems with potential high stiffness.

Traditionally, operator-splitting and time-lagging techniques have been developed to reduce the MGD equations to a series of linear scalar reaction-diffusion problems. It is worthwhile to notice that they predigested the MGD model enough that the medium-sized problems can be effectively solved. However, they would give rise to the demand for unacceptably small time steps in very large-scale simulations and do not scale well when the problem size increases, that is, their inaccuracies and unscalability prevent solution. The fully-implicit (namely, monolithic) solution strategies for the MGD equations have been an active and promising area of research [10,11,31,42]. It is important to point out that the fully-implicit strategy is guaranteed to converge the nonlinearity of the transient solution at each time step [32]. For the purpose of tackling this fully-implicit formulation, we make use of an adaptive backward Euler time-stepping scheme to allow larger time steps without sacrificing accuracy and lessen the total number of timesteps. The method of frozen coefficients [30] we employ is an iterative method based on linearizing the discrete nonlinear problem about the current approximation and then solving multiple systems of linear equations to determine the next approximation. Each of these linear systems comes from a spatial discretization via a cell-centered, locally conservative finite volume scheme with second-order experimental accuracy over the computational grid with the number of mesh cells ranging from 10^6 to 10^{10}. It must be emphasized that these linear systems are all sparse, unsymmetric but positive definite and rather ill-conditioned. It is, therefore, natural to be the most computation- and memory-intensive ingredient, generally making up more than 80% of the simulation period of real-world applications, to provide their computational solutions which are all obedient to certain user-specified accuracy bounds. To deal with such a bottleneck, inefficient algorithms or efficient approaches with difficulties in developing effective parallel implementations would markedly decelerate the extremely large simulations, forcing computational scientists to either simplify the continuous model or tarry too long for the final results.

The existing numerical solution algorithms can be categorized into two different types: sparse direct and iterative methods. The intrinsic appeals of sparse direct solvers,