Commun. Comput. Phys.
doi: 10.4208/cicp.OA-2022-0205

Vol. **33**, No. 3, pp. 647-691
March 2023

Structure-Preserving Finite-Element Schemes for the Euler-Poisson Equations

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Received 23 June 2022; Accepted (in revised version) 16 January 2023

Abstract. We discuss structure-preserving numerical discretizations for repulsive and attractive Euler-Poisson equations that find applications in fluid-plasma and self-gravitation modeling. The scheme is fully discrete and structure preserving in the sense that it maintains a discrete energy law, as well as hyperbolic invariant domain properties, such as positivity of the density and a minimum principle of the specific entropy. A detailed discussion of algorithmic details is given, as well as proofs of the claimed properties. We present computational experiments corroborating our analytical findings and demonstrating the computational capabilities of the scheme.

AMS subject classifications: 65M22, 35L65, 35Q31

Key words: Euler-Poisson equations, operator splitting, invariant domain preservation, discrete energy balance.

1 Introduction

In this manuscript we develop numerical schemes for the repulsive and attractive Euler-Poisson equations. This is a system of equations that combine the hyperbolic compress-

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ible Euler equations of gas dynamics that describe the time evolution of a fluid state (consisting of pressure, momentum and total energy) with the action of a scalar potential that in turn depends on the time evolution of the density of the system. The Euler-Poisson equations have found applications in the context of plasma physics [56], semiconductor device modeling [51], and vacuum electronics [67]. The equations are often used to model an electron fluid subject to electrostatic forces. The Euler-Poisson system is also routinely used in astrophysics [57] for modeling large scale formation of galaxies due to self-gravitation.

Our goal is to develop a numerical method for the Euler-Poisson system that is second order accurate and provably robust. By this we mean that the fully discrete update procedure at each time-step is *locally well-posed*, implying: (a) that the numerical scheme is always able to compute a new *admissible* state (i. e., with positive density and internal energy); (b) it preserves a discrete energy law; and (c) that the linear algebra only involves symmetric positive definite problems, while the time-step size is only subject to a *hyperbolic CFL* condition. Our approach is based on an operator splitting in order to decouple the hyperbolic and elliptic subsystems. We consider a fully-discrete analysis of the scheme, revealing the need of specific choices of space and time discretization that we make precise in Section 3.

The splitting approach allows us to relegate invariant domain preservation entirely to the numerical scheme used for the hyperbolic system; see Section 2. This leaves considerable freedom for the specific choice of hyperbolic solver. In particular, one could choose a numerical method that preserves all invariant sets, or a subset of the invariant set properties, such as positivity of density and internal energy, see [37,42]. The resulting scheme will be invariant domain preserving if the hyperbolic solver preserves all invariants. For the sake of completeness we briefly describe the hyperbolic solver used in this manuscript in Appendix A.

1.1 The Euler-Poisson system

We consider a general model problem derived by coupling the compressible Euler equations of gas dynamics to a scalar potential:

$$\partial_t \rho + \operatorname{div} \mathbf{m} = 0, \tag{1.1a}$$

$$\partial_t \mathbf{m} + \operatorname{div}\left(\rho^{-1}\mathbf{m}\mathbf{m}^\top + Ip\right) = -\rho\nabla\varphi - \frac{1}{\tau}\mathbf{m}, \tag{1.1b}$$

$$\partial_t \mathcal{E} + \operatorname{div}\left(\frac{\mathbf{m}}{\rho}(\mathcal{E}+p)\right) = -\nabla\varphi\cdot\mathbf{m} - \frac{1}{\rho\tau}|\mathbf{m}|^2, \tag{1.1c}$$

$$-\Delta\varphi = \alpha(\rho+\rho_b). \tag{1.1d}$$

Here, $\rho(\boldsymbol{x},t)\in\mathbb{R}^+$ is the mass density, $\mathbf{m}(\boldsymbol{x},t)\in\mathbb{R}^d$, the momentum, $\mathcal{E}(\boldsymbol{x},t)\in\mathbb{R}^+$ the total energy, $p\in\mathbb{R}$ denotes the thermodynamic pressure, and $\rho_b(\boldsymbol{x},t)\in\mathbb{R}$ denotes a prescribed background density that, in contrast to the mass density ρ, might attain negative values. The balance of momentum and total energy equations (1.1b) and (1.1c) include a