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A Decreasing Upper Bound of the Energy for Time-Fractional Phase-Field Equations

Chaoyu Quan[1], Tao Tang[2,3], Boyi Wang[4,5,*] and Jiang Yang[4,6]

[1] *International Center for Mathematics, Southern University of Science and Technology, Shenzhen 518055, China.*
[2] *Division of Science and Technology, BNU-HKBU United International College, Zhuhai 519087, China.*
[3] *Guangdong Provincial Key Laboratory of Interdisciplinary Research and Application for Data Science, BNU-HKBU United International College, Zhuhai 519087, China.*
[4] *Department of Mathematics, Southern University of Science and Technology, Shenzhen 518055, Guangdong, China.*
[5] *Department of Mathematics, National University of Singapore, Singapore 119076, Singapore.*
[6] *Guangdong Provincial Key Laboratory of Computational Science and Material Design, Southern University of Science and Technology, Shenzhen 518055, China.*

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Abstract. In this article, we study the energy dissipation property of time-fractional Allen–Cahn equation. On the continuous level, we propose an upper bound of energy that decreases with respect to time and coincides with the original energy at $t=0$ and as t tends to ∞. This upper bound can also be viewed as a nonlocal-in-time modified energy which is the summation of the original energy and an accumulation term due to the memory effect of time-fractional derivative. In particular, the decrease of the modified energy indicates that the original energy indeed decays w.r.t. time in a small neighborhood at $t=0$. We illustrate the theory mainly with the time-fractional Allen–Cahn equation but it could also be applied to other time-fractional phase-field models such as the Cahn–Hilliard equation. On the discrete level, the decreasing upper bound of energy is useful for proving energy dissipation of numerical schemes. First-order L1 and second-order L2 schemes for the time-fractional Allen–Cahn equation have similar decreasing modified energies, so that stability can be established. Some numerical results are provided to illustrate the behavior of this modified energy and to verify our theoretical results.

AMS subject classifications: 65M06, 65M12, 35R11

Key words: Time-fractional Allen–Cahn equation, energy dissipation, L1 approximation, L2 approximation.

*Corresponding author. *Email addresses:* quancy@sustech.edu.cn (C. Quan), ttang@uic.edu.cn (T. Tang), boyiwang@u.nus.edu (B. Wang), yangj7@sustech.edu.cn (J. Yang)

1 Introduction

Phase-field models have various applications in diverse regions such as multiphase flow [7, 46, 48, 56, 64], material sciences [4, 26, 30, 47], image processing and biology [6, 17, 40], etc. Most phase-fields models are derived as gradient flows associating with some specific energy functional, such as the Ginzburg–Landau energy for Allen–Cahn equations and Cahn–Hilliard equations, Swift–Hohenberg energy for phase-field crystal models. Seeking numerical solutions of phase-field equations has attracted a lot of attentions in the past decade, which could be a delicate task: intrinsic properties of the solution shall be recovered on the discrete level (energy dissipation, maximum principle) and the presence of small parameter $\varepsilon > 0$ can generate practical difficulties. There have been plenty of numerical schemes for phase-field equations, including the convex-splitting schemes [11, 18, 20, 62], the stabilization schemes [55, 63, 66], the implicit-explicit (IMEX) schemes [32, 33, 59], the operator splitting methods [34, 35], the scalar auxiliary variable (SAV) schemes [53, 54], and the exponential time differencing (ETD) schemes [15, 23]. Lately, numerical methods and analysis have also been carried out for time-fractional models which take into account long-time memory effects [10, 28, 37, 38, 65].

Recently, the time-fractional phase-field (TFPF) equations have been considered in different applications. For instance, phase-field framework has been successfully employed to describe the evolution of structural damage and fatigue [9], in which the damage is described by a variable order time fractional derivative. In [41] , the TFPF models account for the anomalously subdiffusive transport behavior in heterogeneous porous materials. The coarsening rate exponents of TFPH equations have also been computationally studied and formally analyzed [10, 41, 58], which agrees with some unusual exponents reported in physics and biology [5, 31]. These problems are challenging due to the existence of both nonlocality and nonlinearity. Since the TFPH equations are extensions of phase-field equations, it is natural to extend the relevant intrinsic properties, i.e., the maximum principle and energy dissipation, to the TFPF equations, e.g., [16, 36, 44].

The Allen–Cahn (AC) model is a popular phase-field model with the governing equation

$$\partial_t u = \gamma(\varepsilon^2 \Delta u - F'(u)), \quad (t,x) \in (0,T) \times \Omega, \tag{1.1}$$

where $\varepsilon > 0$ is the interface width, $\gamma > 0$ is the diffusion mobility constant, and $F(u)$ is a potential function. For example, a simple and popular choice is the double well potential

$$F(u) = \frac{1}{4}(u^2 - 1)^2. \tag{1.2}$$

For other types of potential function, we refer to [13, 14]. The energy functional of the AC equation (1.5) is

$$E(t) := \int_\Omega \left(\frac{\varepsilon^2}{2} |\nabla u|^2 + F(u) \right) \mathrm{d}x. \tag{1.3}$$