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COMPUTING HARMONIC MAPS AND CONFORMAL MAPS ON POINT CLOUDS*

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Abstract

We use a narrow-band approach to compute harmonic maps and conformal maps for surfaces embedded in the Euclidean 3-space, using point cloud data only. Given a surface, or a point cloud approximation, we simply use the standard cubic lattice to approximate its ϵ-neighborhood. Then the harmonic map of the surface can be approximated by discrete harmonic maps on lattices. The conformal map, or the surface uniformization, is achieved by minimizing the Dirichlet energy of the harmonic map while deforming the target surface of constant curvature. We propose algorithms and numerical examples for closed surfaces and topological disks. To the best of the authors' knowledge, our approach provides the first meshless method for computing harmonic maps and uniformizations of higher genus surfaces.

Mathematics subject classification: 68U05.
Key words: Harmonic maps, Conformal maps, Point clouds.

1. Introduction

Roughly speaking, a map between two surfaces is called *conformal* if it preserves angles, and is called *harmonic* if it minimizes the stretching energy. Computing harmonic maps and conformal maps has a wide range of applications, such as surface matching, surface parameterization, shape analysis and so on. See [1–11] for examples of applications of conformal maps, and [12–16] for examples of applications of harmonic maps.

Existing methods for computing harmonic maps and conformal maps mostly rely on the triangle mesh approximation of a surface. However, it is often much easier to get point cloud data, rather than the triangle mesh data. Contemporary 3D scanners can easily provide 3D point cloud data sampled from the surfaces of solid objects, but sometimes it is inconvenient to generate meshes upon point clouds. Since point clouds data do not contain information about the connectivity, a lot of algorithms, which were well-established on meshes, cannot be extended to point clouds directly.

Various meshless methods have been developed for computing conformal maps, mainly for topological spheres and topological disks. One of the most used idea is to approximate the Laplace-Beltrami operators or related differential equations on point clouds.

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This paper implemented a narrow-band approach. The idea of narrow bands was first used by Sethian et al. [17] in the numerical implementation of the level-set method developed by Osher et al. [18]. Later the idea was extended to computing eigenvalues of elliptic operators on surfaces by Brandman [19] and Gao et al. [20].

We implement the idea of narrow-bands in computing harmonic maps and conformal maps for a surface in the Euclidean 3-dimensional space. The basic idea is to use a dense 3-dimensional lattice to approximate the ϵ-neighborhood of the surface, and then compute the discrete harmonic map from the lattice to the target surface, by minimizing the Dirichlet energy (i.e., the stretching energy). Conformal maps, or surface uniformizations, are computed by minimizing the Dirichlet energy of the harmonic maps, as we deform the target surface of constant Gaussian curvature. In this paper, we focus on harmonic diffeomorphisms and conformal diffeomorphisms to surfaces of constant curvature ± 1 or 0. These maps are particularly useful for global surface parameterizations. More specifically, we propose algorithms and numerical examples for (1) maps to the unit sphere, and (2) maps to flat rectangles, and (3) maps to flat tori, and (4) maps to closed hyperbolic surfaces.

Not like the other meshless methods mentioned above, we do not approximate the Laplace-Beltrami operator or related differential equations on the given point clouds. But we do approximate the harmonic maps on a surface using harmonic maps on its ϵ-neighborhood, which is approximated using cubic lattice discretizations.

1.1. Previous Works

Comparing to methods for triangle meshes, there are much fewer works on meshless methods of computing conformal and harmonic maps, especially for higher genus surfaces. Guo et al. [21] computed global conformal parameterizations of surfaces by computing holomorphic 1-forms on point clouds. Li et al. [22] computed harmonic volumetric maps by grids discretizations. Meng-Lui [23] developed the theory of computational quasiconformal geometry on point clouds. Using approximations of the differential operators on point clouds, Liang et al. [24,25] and Choi et al. [26] computed the spherical conformal parameterizations of genus-0 closed surfaces, and Meng et al. [27] computed quasiconformal maps on topological disks. Li-Shi-Sun [28] computed quasiconformal maps from surfaces to planar domain, using the so-called point integral method for discretizing integral equations for point clouds. Liu et al. [29] developed a free-boundary conformal parameterization method for disk-type point clouds, where the geometric distortion is much less than usual fixed-boundary methods. Other methods to approximate the Laplace-Beltrami operator on point clouds can also be found in [30,31].

There is an extensive literature on computing conformal maps for triangle meshes. Gu-Yau [32,33] developed the method of computing conformal structures of surfaces by computing the discrete holomorphic one-forms. Pinkall-Polthier [13] proposed a method of conformal parameterization by computing a pair of conjugate harmonic functions. Lévy et al. [34] and Lipman [35] and Lui et al. [36] computed conformal or quasiconformal maps by minimizing or controlling the conformal distortion. There is also a big family of methods based on various notions of discrete conformality for triangle meshes, such as circle patterns [37–40], and inversive distances [41,42], and vertex scalings [40,43,44], and modified vertex scalings [45–48] allowing diagonal switches. Some related convergence results for discrete conformality can be found in [37,49–52], and other mathematical analysis can be found in [53–57]. Other works on computing conformal maps on triangle meshes include [34,58–74].

For computing harmonic maps on triangle meshes, Gaster-Loustau-Monsaingeon [75,76] give