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Linearized Learning with Multiscale Deep Neural Networks for Stationary Navier-Stokes Equations with Oscillatory Solutions

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Congratulations on the 60th birthday of Professor Tao Tang with friendship and admiration for his many achievements!

Abstract. In this paper, we present linearized learning methods to accelerate the convergence of training for stationary nonlinear Navier-Stokes (NS) equations. To solve the stationary nonlinear NS equation with deep neural networks, we integrate linearizations of the nonlinear convection term in the NS equations into the training process of multi-scale deep neural network (DNN) approximations of the NS solution. Four forms of linearizations are considered. We solve highly oscillating stationary flows in complex domains utilizing the proposed linearized learning with multiscale neural networks. The results show that multiscale deep neural network combined with the linearized schemes can be trained much faster and accurately than regular fully connected DNNs.

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Key words: Linearized learning, stationary Navier-Stokes equation, multi-scale deep neural network.

1. Introduction

Deep neural network (DNN) machine learning methods have been researched as alternative numerical methods for solving partial differential equations arising from many practical engineering problems. The deep learning framework for solving those kinds of problems uses the given partial differential equations as regularization in the loss function during training, where the auto-differentiation can be applied to the inputs of the neural network. Since auto differentiation with respect to the inputs of neural network are built-in, thus there is no need for any pre-generated meshes in the solution domain. Therefore,

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such a framework has the potential of being a flexible meshless method to solve governing equations from fluid and solid mechanics in complex geometries, as an alternative method to traditional finite element method. Moreover, these methods have shown much power in solving high dimensional parabolic PDEs [6, 12, 18].

Fluid mechanics, on the other hand, has also been one of the active research fields for the applications of neural network with physical information as regularizations. In the work of [2, 13], the authors proposed a method that combines the Navier-Stokes (NS) equation with visualization data to predict the velocity field and pressure field, with synthetic data in [13] and real experimental imaging data in [2], respectively. In [5], a physical-informed neural network is used for solving the Reynolds-averaged Navier-Stokes equations with Reynolds-stress components ($\overline{u^2}, \overline{uv}$, and $\overline{v^2}$) as extra outputs of the neural networks. Rao *et al.* [14] proposed a mixed-variable scheme with Cauchy stress tensor to eliminate the intractability of the complex form of naive Navier-Stokes equation and its high-order derivatives (e.g., ∇^2) and this scheme was applied to learn the steady flow and the transient flow passing a cylinder respectively. Furthermore, Oldenburg *et al.* [11] proposed the Geometry Aware Physics Informed Neural Network to handle the Navier-Stokes equations with irregular geometry where they utilize the shape encoding network, i.e., an encoder, to reduce the geometry dimensions to a size-fixed latent vector k and k will be the input of two additional neural networks, one to handle the boundary constraints and one to handle physical information, i.e., the governing PDEs. In the meantime, the error estimations for neural networks to approximate the Navier-Stokes equations has been studied in [4].

Recent studies on DNNs have shown that they have a frequency dependence performance in learning solution of PDEs and fitting functions. Namely, the lower frequency components of the solution are learned first and quickly compared with the higher frequency components [17]. Several attempts have been made to remove such a frequency bias for the DNNs. The main idea is to convert the higher frequency content of the solution to a lower frequency range so the conventional DNNs can learn the solution in acceptable training epochs. One way to achieve this goal is to use phase shifts [3] while the other is to introduce a multiscale structure into the DNNs [10] where in which sub-neural networks with different scales will target different ranges of the frequency in the solutions. The PhaseDNN has been shown to be very effective for high frequency wave propagation while the MscaleDNN [10] has been used to learn highly oscillatory Stokes flow solutions in complex domains [16] as well as high dimensional PDEs [18].

Most of the previous works are focusing on Linear PDEs. The learning of the solution of linear PDEs via least squared residuals of the PDEs is in some sense equivalent to a fitting problem in the frequency domain in view of the Parseval's identity of Fourier transforms. So it is natural the performance improvements of multiscale DNN also holds for learning the solution of linear PDEs.

Additional difficulties arise when there are nonlinearities introduced in the PDEs. Based on the results from Jin *et al.* [8], it is found that it could take $\mathcal{O}(10^4)$ epochs to solve a simple domain problem, thus ineffective and impractical especially when highly oscillating problems are to be considered. Also, the MscaleDNN applied directly to the nonlinear Navier-Stokes equation did not produce the same large improvement over conventional DNNs as