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GAUGE-UZAWA METHODS FOR THE NAVIER-STOKES EQUATIONS WITH NONLINEAR SLIP BOUNDARY CONDITIONS

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Abstract. In this paper, the Gauge-Uzawa method is applied to solve the Navier-Stokes equations with nonlinear slip boundary conditions whose variational formulation is a variational inequality of the second kind with the Navier-Stokes operator. In [1], a multiplier was introduced such that the variational inequality is equivalent to the variational identity. We give the Gauge-Uzawa scheme to compute this variational identity and provide a finite element approximation for the Gauge-Uzawa scheme. The stability of the Gauge-Uzawa scheme is showed. Finally, numerical experiments are given, which confirm the theoretical analysis and demonstrate the efficiency of the new method.

Key words. Navier-Stokes equations, nonlinear slip boundary, variational inequality, Gauge-Uzawa method, finite element approximation.

1. Introduction

Numerical simulation for incompressible flow is a fundamental and significant problem in computational mathematics and computational fluid mechanics. It is well known that a mathematical model for a viscous incompressible fluid with homogeneous boundary conditions involves the Navier-Stokes equations.

In this paper, we will consider the following Navier-Stokes equations

$$(1)\qquad \begin{cases} \dfrac{\partial u}{\partial t} - \mu\Delta u + (u\cdot\nabla)u + \nabla p = f, \\ \mathrm{div} u = 0. \end{cases}$$

It is obvious that (1) is a coupled system with a first-order nonlinear evolution equation and an imposed incompressible constrain so that the numerical simulation for the Navier-Stokes equations is very difficult. A popular technique to overcome this difficulty is to relax the solenoidal condition in an appropriate method to result in a pesudo-compressible system, such as a penalty method or a artificial compressible method. An operator splitting method is also very useful to overcome this shortage. The main advantage of the operator splitting method is that it can decouple the difficulties associated to the nonlinear property with those associated to the incompressible condition. For more details, see [2].

The Gauge-Uzawa method has been a popular tool for the numerical simulation of incompressible viscous flow. The purpose of this paper is to propose two new Gauge-Uzawa schemes for incompressible flows with nonlinear slip boundary conditions. This class of boundary conditions are introduced by Fujita in [3, 4, 5]. The first scheme will be based on a system in convected form [6] while the second scheme will be based on the stabilized Gauge-Uzawa method [7]. We recall that the

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Gauge-Uzawa method is introduced in [6, 8] to overcome some implementation difficulties associated with the Gauge method introduced in [9]. It has been shown in [6, 10, 11, 12] that the Gauge-Uzawa method has many advantages over the original Gauge method and the pressure-correction method. We will show that a proper Gauge-Uzawa formulation is well suitable for problems with variable density. More precisely, our two new schemes will only involve one projection step and will be proved unconditionally stable.

The paper is organized as follows. In the next two sections, we present the two Gauge-Uzawa schemes and show that they are unconditionally stable, respectively. In Section 4, we decrible the finite element approximation of the two Gauge-Uzawa schemes. In Section 5, we present some numerical results which reveal the convergence rate of our schemes for each of the three unknown functions and some concluding remarks are given.

2. Navier-Stokes Equations with Nonlinear Slip Boundary Conditions

Consider the Navier-Stokes equations:

$$(2)\qquad \begin{cases} \dfrac{\partial u}{\partial t} - \mu\Delta u + (u\cdot\nabla)u + \nabla p = f & \text{in } Q_T, \\ \text{div} u = 0 & \text{in } Q_T, \end{cases}$$

where $Q_T = \Omega\times[0,T]$ for some $T > 0$, $u(t,x)$ denotes velocity, $p(t,x)$ denotes pressure, and $f(t,x)$ denotes the external force. The domain $\Omega\subset R^2$ is a bounded domain. Given the initial value $u(0,x) = u_0(x)$ in Ω, we consider the following nonlinear slip boundary conditions:

$$(3)\qquad \begin{cases} u = 0, & \text{on } \Gamma, \\ u_n = 0, \quad -\sigma_\tau(u)\in g\partial|u_\tau| & \text{on } S, \end{cases}$$

where $\Gamma\cap S = \emptyset, \overline{\Gamma\cup S} = \partial\Omega$ with $|\Gamma|\neq 0, |S|\neq 0$. The viscous coefficient $\mu > 0$ is a positive constant, g is a scalar function, and $u_n = u\cdot n$ and $u_\tau = u - u_n n$ are the normal and tangential components of the velocity, where n stands for the unit vector of the external normal to S. $\sigma_\tau(u) = \sigma - \sigma_n n$, independent of p, is the tangential components of the stress vector σ which is defined by $\sigma_i = \sigma_i(u,p) = (\mu e_{ij}(u) - p\delta_{ij})n_j$, where $e_{ij}(u) = \frac{\partial u_i}{\partial x^j} + \frac{\partial u_j}{\partial x^i}, i,j = 1,2$. The set $\partial\psi(a)$ denotes a subdifferential of the function ψ at the point a:

$$\partial\psi(a) = \{b\in R^2 : \psi(h) - \psi(a) \geq b\cdot(h-a) \quad \forall\ h\in R^2\}.$$

Introduce

$$V = \{u\in H^1(\Omega)^2,\ u|_\Gamma = 0,\ u\cdot n|_S = 0\},\quad V_0 = H^1_0(\Omega)^2,$$

$$V_\sigma = \{u\in V,\quad \text{div} u = 0\},\qquad M = L^2_0(\Omega) = \{q\in L^2(\Omega), (1,q)_{L^2(\Omega)} = 0\}.$$

Let $||\cdot||_k$ be the norm in the Hilbert space $H^k(\Omega)^2$, and $(\cdot,\cdot)$ and $||\cdot||$ be the inner product and the norm in $L^2(\Omega)^2$, respectively. Then we can equip the inner product and the norm in V by $(\nabla\cdot,\nabla\cdot)$ and $||\cdot||_V = ||\nabla\cdot||$, respectively, because $||\nabla\cdot||$ is equivalent to $||\cdot||_1$. Let $\mathbb{X}$ be a Banach space. Denote by $\mathbb{X}'$ the dual space of $\mathbb{X}$ and $<\cdot,\cdot>$ be the dual pairing in $\mathbb{X}\times\mathbb{X}'$. Also we will use δ as a difference of two functions, for example, for any sequence function z^{n+1},

$$\delta z^{n+1} = z^{n+1} - z^n,\ \delta\delta z^{n+1} = \delta(\delta z^{n+1}) = z^{n+1} - 2z^n + z^{n-1}, \cdots .$$