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Congruences Involving Hecke-Rogers Type Series and Modular Forms

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Abstract. In this paper, we prove two supercongruences of Hecke-Rogers type series and Modular forms conjectured by Chan, Cooper and Sica, such as, if

$$z_2 = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} q^{m^2+n^2}, \quad x_2 = \frac{\eta^{12}(2\tau)}{z_2^6}$$

and

$$z_2 = \sum_{n=0}^{\infty} f_{2,n} x_2^n,$$

then

$$f_{2,pn} \equiv f_{2,n} \pmod{p^2} \text{ when } p \equiv 1 \pmod 4,$$

where

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^n),$$

and $q = exp(2\pi i \tau)$ with $Im(\tau) > 0$.

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1 Introduction

Hecke-Rogers type series are of the following type:

$$\sum_{(m,n)\in D} (-1)^{H(m,n)} q^{Q(m,n)+L(m,n)},$$

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where H and L are linear forms, Q is a quadratic form, and D is some subset of $\mathbb{Z}\times\mathbb{Z}$. The classical identity of Jacobi is of this type:

$$\sum_{n=-\infty}^{\infty}\sum_{m\geq|n|}(-1)^m q^{(m^2+m)/2}=\prod_{n=1}^{\infty}(1-q^n)^3.$$

Motivated by the Jacobi identity, Hecke [7] investigated theta functions related to indefinite quadratic forms systematically. For example, Hecke [7, p. 425] found that

$$\sum_{n=-\infty}^{\infty}\sum_{|m|\leq n/2}(-1)^{n+m} q^{(n^2-3m^2)/2+(n+m)/2}=\prod_{n=1}^{\infty}(1-q^n)^2,$$

which is originally due to Rogers [13, p.323].

In his proof of the irrationality of $\zeta(3)$, Apéry [2] introduced the numbers

$$A_n=\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}^2, \quad n\in\mathbb{N}=\{0,1,\dots\}.$$

These numbers are now known as the Apéry numbers. The properties of A_n are gradually investigated since the work of Apéry was appeared. One of the properties is that for primes $p\geq 5$,

$$A_p\equiv A_1 \pmod{p^3}.$$

This congruence was conjectured by Chowla *et al.* [3] and proved by Gessel [6], who established the stronger result

$$A_{pn}\equiv A_n \pmod{p^3}.$$

Peters and Stienstra [12] showed that if

$$G(z)=\frac{\eta^7(2z)\eta^7(3z)}{\eta^5(z)\eta^5(6z)} \quad \text{and} \quad s(z)=\left(\frac{\eta(6z)\eta(z)}{\eta(2z)\eta(3z)}\right)^{12},$$

then we have

$$G(z)=\sum_{n=0}^{\infty}A_n s^n(z),$$

where

$$\eta(\tau)=q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n),$$

and $q=exp(2\pi i\tau)$ with $Im(\tau)>0$.

About the modular forms, the reader may consult [10]. Osbrun *et al.* also got some supercongruences for Apéry-like numbers.

Motivated by work of [6,12], Chan *et al.* [4] proved the following theorem