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# Exponential Convergence Theory of the Multipole and Local Expansions for the 3-D Laplace Equation in Layered Media

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Dedicated to the memory of Professor Zhongci Shi

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**Abstract.** In this paper, we establish the exponential convergence theory for the multipole and local expansions, shifting and translation operators for the Green's function of 3-dimensional Laplace equation in layered media. An immediate application of the theory is to ensure the exponential convergence of the FMM which has been shown by the numerical results reported in [27]. As the Green's function in layered media consists of free space and reaction field components and the theory for the free space components is well known, this paper will focus on the analysis for the reaction components. We first prove that the density functions in the integral representations of the reaction components are analytic and bounded in the right half complex wave number plane. Then, by using the Cagniard-de Hoop transform and contour deformations, estimates for the remainder terms of the truncated expansions are given, and, as a result, the exponential convergence for the expansions and translation operators is proven.

**AMS subject classifications**: 15A15, 15A09, 15A23

**Key words**: Fast multipole method, layered media, multipole expansions, local expansions, 3-D Laplace equation, Cagniard–de Hoop transform, equivalent polarization sources.

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# 1 Introduction

Many important computational problems in science and engineering involve solving the Laplace equation in layered media. For instance, finding the electric static potential in a layered dielectric medium has important application in semi-conductor industry, such as calculating the capacitance of interconnects (ICs) in very large-scale integrated (VLSI) circuits for microchip designs [22–24, 31]. Other applications of solving Laplace equation in layered media can be found in medical imaging of brains [30], modeling of triboelectric nanogenerators [19], elasticity of composite materials [3–5], complex scattering problem in meta-materials [8], electrostatic potential computation in ion channel simulation [20], and electrical impedance tomography for geophysical applications [6].

Due to complex geometric structure of the physical objects and the layered medium setting from aforementioned application problems, integral methods with the Green's function of layered media (cf. [22, 33]) are usually adopted, which results in a huge dense linear algebraic system to be solved by an iterative method such as GMRES [7], etc. As a result, it will incur an $\mathcal{O}(N^2)$ computational cost for computing the product of a $N\times N$ matrix with a vector (a basic operation for the GMRES iterative solver). The well-known fast multipole method (FMM) proposed by Greengard and Rohklin [13, 14] for sources in free space has been applied to accelerate the iterative solvers for dense linear system resulting from boundary integral methods [21, 25]. However, the algorithm is only applicable for problems in free space and the FMM for Green's function in layered media has been one of the important un-resolved problems in the fast algorithm research community. Since the early 1990s, many researchers have been working on this problem and proposed several fast algorithms including complex image approximation [2, 10, 12, 16, 18], inhomogeneous plane wave expansion [9, 17], etc. Nevertheless, the multipole expansion theory of the Green's function of Laplace equation in layered media has not been established, which forms the core component of the FMMs.

The free space FMM was based on low rank approximations for the far field of sources, obtained by using truncated multipole expansions (MEs) and local expansions (LEs) with a small truncation number $p$. The capability of using a small number $p$ to achieve high accuracy is due to the exponential convergence of the MEs and LEs, as well as the shifting and translation operators for multipole to multipole (M2M), local to local (L2L), and multipole to local (M2L) conversions. The mathematical foundation of the MEs and their shifting and translation operators is the classical addition theory for Legendre polynomials or Bessel functions (cf. [1, 11, 15, 29]).

Recently, we have derived MEs, LEs and translation operators for Green's func-