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Blow-Up and Boundedness in Quasilinear Parabolic-Elliptic Chemotaxis System with Nonlinear Signal Production

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Abstract. In this paper, we consider the quasilinear chemotaxis system of parabolic-elliptic type

$$\begin{cases} u_t = \nabla\cdot(D(u)\nabla u) - \nabla\cdot(f(u)\nabla v), & x\in\Omega,\ t>0, \\ 0 = \Delta v - \mu(t) + g(u), & x\in\Omega,\ t>0 \end{cases}$$

under homogeneous Neumann boundary conditions in a smooth bounded domain $\Omega\subset\mathbb{R}^n$, $n\geq 1$. The nonlinear diffusivity $D(\xi)$ and chemosensitivity $f(\xi)$ as well as nonlinear signal production $g(\xi)$ are supposed to extend the prototypes

$$D(\xi)=C_0(1+\xi)^{-m},\ f(\xi)=K(1+\xi)^k,\ g(\xi)=L(1+\xi)^l,\ C_0>0,\xi\geq 0,K,k,L,l>0,m\in\mathbb{R}.$$

We proved that if $m+k+l>1+\frac{2}{n}$, then there exists nonnegative radially symmetric initial data u_0 such that the corresponding solutions blow up in finite time. However, the system admits a global bounded classical solution for arbitrary initial datum when $m+k+l<1+\frac{2}{n}$.

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1 Introduction

We study the initial-boundary value problem

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$$\begin{cases} u_t = \nabla\cdot(D(u)\nabla u) - \nabla\cdot(f(u)\nabla v), & x\in\Omega,\ t>0, \\ 0 = \Delta v - \mu(t) + g(u), & x\in\Omega,\ t>0, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & x\in\partial\Omega,\ t>0, \\ u(x,0) = u_0(x), & x\in\Omega \end{cases} \tag{1.1}$$

in a bounded domain $\Omega\subset\mathbb{R}^n, n\geq 1$ with smooth boundary, where $\frac{\partial}{\partial\nu}$ denotes outward normal derivatives on $\partial\Omega$, f and g are nonnegative Hölder continuous functions, and $\mu(t) = \frac{1}{|\Omega|}\int_\Omega g(u(x,t))\mathrm{d}x$ is the time-dependent spatial mean of signal nonlinearity $g(u)$.

Chemotaxis is the movement of biological organisms oriented towards the gradient of some chemical signal substance. This phenomenon can be described by the Keller-Segel model, initiated by celebrated work [1]. Therefore, let's look at the simplest version of the parabolic Keller-Segel model as follows

$$\begin{cases} u_t = \Delta u - \nabla\cdot(u\nabla v), & x\in\Omega,\ t>0, \\ v_t = \Delta v - v + u, & x\in\Omega,\ t>0, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & x\in\partial\Omega,\ t>0, \\ u(x,0) = u_0(x), v(x,0) = v_0(x), & x\in\Omega, \end{cases} \tag{1.2}$$

where $\Omega\subset\mathbb{R}^n$ is a smooth boundary, u denotes the cell density and v represents the concentration of chemical signal. The initial distributions u_0 and v_0 are nonnegative functions. In the past three decades, the system (1.2) has been investigated quite extensively on the existence of global bounded solutions or focused on the occurrence of blow-up in finite time. When $n=1$, all solutions of (1.2) are global bounded in [2]. The solution would be global in time and bounded in [3,4] or the finite-time blowup is derived in [5,6] for $n=2$ with large initial mass and in [7,8] for $n\geq 3$. When the cell's movement is much slower than the chemical signal diffusing, the second equation in (1.2) could be simplified to

$$0 = \Delta v - \mu(t) + u,$$

where $\mu(t) := \frac{1}{|\Omega|}\int_\Omega u(\cdot,t)\mathrm{d}x > 0$, and the simplified system was proved to have a finite-time blow-up phenomenon in [9] when $n=2$ and in [10] for $n\geq 2$. Next, let's look at one called volume-filling effect in the quasilinear Keller-Segel system in [11], stated as follows

$$\begin{cases} u_t = \nabla\cdot(D(u)\nabla u) - \nabla\cdot(uS(u)\nabla v), & x\in\Omega,\ t>0, \\ v_t = \Delta v - v + u, & x\in\Omega,\ t>0 \end{cases} \tag{1.3}$$

subject to homogeneous Neumann boundary conditions, where the functions $D(u)$ and $S(u)$ denote the strength of diffusion and chemoattractant, respectively. Subsequently, Tao and Winkler in [12] proved that the classical solution of (1.3) is uniformly-in-time