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Regularity Criteria of the Magnetohydrodynamic Equations in a Bounded Domain

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Abstract. Regularity criteria in terms of bounds for the pressure are derived for the 3D MHD equations in a bounded domain with slip boundary conditions. A list of three regularity criteria is shown.

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1 Introduction and main results

We consider the initial-boundary value problem of three-dimensional incompressible magnetohydrodynamic (MHD) equations (see [1]):

$$\operatorname{div} u = \operatorname{div} b = 0, \tag{1.1}$$

$$\partial_t u + (u\cdot\nabla)u + \nabla\left(\pi + \frac{1}{2}|b|^2\right) - \mu\Delta u = (b\cdot\nabla)b, \tag{1.2}$$

$$\partial_t b + u\cdot\nabla b - b\cdot\nabla u - \eta\Delta b = 0, \tag{1.3}$$

in $Q_T := \Omega\times[0,T)$ with slip boundary conditions:

$$u\cdot\nu = 0, \quad \operatorname{curl} u\times\nu = 0 \quad \text{ on } \partial\Omega, \tag{1.4}$$

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$$b\cdot\nu=0,\quad \mathrm{curl} b\times\nu=0 \qquad \text{on } \partial\Omega, \tag{1.5}$$

and initial data

$$(u,b)(x,0)=(u_0,b_0)(x) \ \text{ in } \Omega, \tag{1.6}$$

where u is the velocity field, b is the magnetic field and π is the pressure. $\Omega\subset\mathbb{R}^3$ is a bounded domain with smooth boundary, and ν is the unit outward normal vector along boundary $\partial\Omega$. The parameter $\mu>0$ denotes the viscous coefficient and $\eta>0$ is the resistivity coefficient.

The present paper is mainly concerned with the regularity of solutions to the problem (1.1)-(1.6). When $\Omega:=\mathbb{R}^3$, various regularity criteria for the system (1.1)-(1.3) have been obtained in [2–7]. When Ω is a bounded domain in $\mathbb{R}^3$, the following type of regularity criterion for (1.1)-(1.6) has been proved by Kang-Kim [8] and Fan-Li-Nakamura-Tan [9]:

$$u\in L^s(0,T;L^r(\Omega)) \ \text{ with } \ \frac{2}{s}+\frac{3}{r}=1,\ 3<r\le\infty. \tag{1.7}$$

Our purpose is to present another regularity criteria for the problem (1.1)-(1.6) in terms of pressure. The main result of the present paper is given in the following theorem.

Theorem 1.1. *Let* $u_0,b_0\in W^{1,3}(\Omega)$ *with* $\mathrm{div}u_0=\mathrm{div}b_0=0$ *in* Ω *and* $u_0\cdot\nu=b_0\cdot\nu=0$ *on* $\partial\Omega$. *Let* (u,b) *be the local strong solution to the problem* (1.1)-(1.6). *If* $p:=\pi+\frac{1}{2}|b|^2$ *satisfies one of the following three conditions:*

$$(i)\ \int_0^T \frac{\|p(t)\|_{L^r(\Omega)}^{\frac{2r}{2r-3}}}{1+\log(e+\|p(t)\|_{L^r(\Omega)})}\mathrm{d}t<\infty \ \text{ with } \ \frac{3}{2}<r<\infty, \tag{1.8}$$

$$(ii)\ \int_0^T \frac{\|p(t)\|_{BMO(\Omega)}}{1+\log(e+\|p(t)\|_{L^r(\Omega)})}\mathrm{d}t<\infty \ \text{ with } \ 1<r<\infty, \tag{1.9}$$

$$(iii)\ \nabla p\in L^{\frac{2r}{3r-3}}(0,T;L^r(\Omega)) \ \text{ with } \ 1<r<\infty, \tag{1.10}$$

with $0<T<\infty$, *then the solution* (u,b) *can be extended beyond* $T>0$. *Here* BMO *is the space of bounded mean oscillation.*

Remark 1.1. The referee informed us that Tran and Yu (J. Math. Phys. provisionally accepted) have derived the following criterion

$$\int_0^T \frac{\|p\|_{L^r}^s}{(1+\|u\|_{L^3})^\kappa}\mathrm{d}t<\infty,$$

for NS. Here $\kappa=3$ for $r\in\left(\dfrac{3}{2},\dfrac{9}{4}\right)$ and becomes smaller for greater r. It seems a similar criterion is possible for MHD.