Conservative Three-Level Linearized Finite Difference Schemes for the Fisher Equation and Its Maximum Error Estimates

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Abstract. A three-level linearized difference scheme for solving the Fisher equation is firstly proposed in this work. It has the good property of discrete conservative energy. By the discrete energy analysis and mathematical induction method, it is proved to be uniquely solvable and unconditionally convergent with the second-order accuracy in both time and space. Then another three-level linearized compact difference scheme is derived along with its discrete energy conservation law, unique solvability and unconditional convergence of order two in time and four in space. The resultant schemes preserve the maximum bound principle. The analysis techniques for convergence used in this paper also work for the Euler scheme, the Crank-Nicolson scheme and others. Numerical experiments are carried out to verify the computational efficiency, conservative law and the maximum bound principle of the proposed difference schemes.

AMS subject classifications: 65M06, 65M12, 65M15

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1. Introduction

The Fisher equation belongs to the class of reaction-diffusion equation: in fact, it is one of the simplest semilinear reaction-diffusion equations, the one which has the inhomogeneous term

$$f(u, x, t) = \lambda u(1 - u),$$

which can exhibit traveling wave solutions that switch between equilibrium states given by $f(u) = 0$. Such equation describes a balance between linear diffusion and nonlinear

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reaction, and it occurs, e.g., in ecology, biology, physiology, combustion, crystallization, plasma physics, and in general phase transition problems. Fisher proposed this equation in 1937 to describe the spatial spread of an advantageous allele and explored its travelling wave solutions [6]. In the same year as Fisher, Kolmogorov *et al.* introduced a more general reaction-diffusion equation [9].

The wider use of this equation in many applications of engineering has been found by researchers. There have been many numerical and approximate methods in the literature to solve this equation, such as the finite difference method, the collocation method, the finite element technique, the wavelet Galerkin method, the pseudospectral method, the various differentiation quadrature method and so on. Here we mainly recall some relevant discretizations based on finite differences. In 1985, Aggarwal [1] compared various difference numerical methods for solving the Fisher equation, including the standard implicit, the quasi linear implicit, the time-linearization implicit, the Crank-Nicolson implicit, the predictor-corrector explicit and two forms of operator-splitting schemes, by the technique of plotting an optimized error-norm versus CPU time. The conclusion that the two-step operator splitting procedure is the most effective method has been drawn. A highly accurate finite difference approach for the second-order spatial derivative in conjunction with a TVD-RK3 method in time was presented to solve the Fisher equation in [2], but there was no any theoretical analysis on the derived scheme. Hasnain *et al.* [8] discussed three difference schemes for solving the Fisher equation: the forward Euler central space scheme, the Lax Wendroff central space scheme and the nonlinear Crank-Nicolson scheme, then the Richardson extrapolation technique was used to improve the numerical accuracy. The Neumann stability analysis was made for the linear form of the resultant difference equation. Chandraker *et al.* [3] proposed two implicit difference schemes to solve the Fisher equation: one is the modified Crank-Nicolson scheme and the other one is the modified Keller box scheme, where the nonlinearity is handled by the method of lagging. The accuracy and stability of the proposed schemes are both discussed based on the numerical experiments.

The considered equation (1.1) is a semilinear parabolic equation and satisfies the maximum principle, or say the maximum bound principle (MBP), i.e., the solution has the range in the set $[0, 1]$ at any time if the initial and boundary values have the same property. Such a problem has been discussed under a systematical framework in [5] along with some provable MBP-preserving numerical schemes. It is always expected that discrete numerical formats have this property. The authors in [10] pointed out that there are few works to study the capability of the numerical methods for solving the Fisher equation to preserve the structure of solutions although abundant numerical schemes can be found in the literature. They proposed a finite difference scheme in a logarithmic form based on the logarithmic form of the continuous model and showed that the scheme can preserve the positivity, the boundedness and the monotonicity of the numerical approximations. The accuracy is of order 1 in time and order 2 in space. Sun *et al.* [14] constructed several difference schemes to solve the Fisher equation and analyzed some conditions to preserve the boundedness and monotone