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MC-Nonlocal-PINNs: Handling Nonlocal Operators in PINNs Via Monte Carlo Sampling

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Abstract. We propose Monte Carlo Nonlocal physics-informed neural networks (MC-Nonlocal-PINNs), which are a generalization of MC-fPINNs in L. Guo *et al.* (Comput. Methods Appl. Mech. Eng. 400 (2022), 115523) for solving general nonlocal models such as integral equations and nonlocal PDEs. Similar to MC-fPINNs, our MC-Nonlocal-PINNs handle nonlocal operators in a Monte Carlo way, resulting in a very stable approach for high dimensional problems. We present a variety of test problems, including high dimensional Volterra type integral equations, hypersingular integral equations and nonlocal PDEs, to demonstrate the effectiveness of our approach.

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1. Introduction

Deep neural networks have gained a growing interest in recent years with a wide variety of methods ranging from computer vision and natural language processing to simulations of physical systems [9, 10, 18, 27]. A representative example is physics-informed neural networks (PINNs) [23], whose central idea is to incorporate governing laws of physical systems into the training loss function and recast the original problem into an optimization problem. PINNs have demonstrated remarkable success in applications including fluid mechanics [3, 24], high dimensional PDEs (with applications in computational finance) [14, 15, 33], uncertainty quantification [13, 16, 19, 22, 31, 34], to name just a few.

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For PDE models with classic (integer) derivatives, PINNs adopt automatic differentiation to solve PDEs by penalizing the PDE in the loss function at a random set of points in the domain of interest. However, for PDE models involving nonlocal operators, one can no longer use automatic differentiation to handle the operators due to the nonlocal property. To overcome this issue, fPINNs [21] were developed for solving space-time fractional advection-diffusion equations. The main idea in [21] is to introduce a classic discretization technique to handle the fractional operator. However, this is not a good choice for high dimensional problems since the curse of dimensionality. Similar idea has been used to handle more general non-local operators in [20], while the approach again can not be used for high dimensional cases. We also mention the work [32], where so-called A-PINN was proposed to handle some special types of integral equations.

More recently, the MC-fPINNs approach was proposed in [12] to handle fractional PDEs, where the fractional operators are handled in a Monte Carlo way, resulting in a very stable approach for high dimensional problems. Take the fractional Laplacian equation as an example

$$(-\Delta)^{\frac{\alpha}{2}}u(x) = C_{d,\alpha}\ \text{P.V.}\int_{\mathbb{R}^d}\frac{u(x)-u(y)}{\|x-y\|_2^{d+\alpha}}dy, \quad 0<\alpha<2, \tag{1.1}$$

where P.V. denotes the principle value of the integral and $C_{d,\alpha}$ is a constant depending on α and d. One can divide the integral into the following two parts:

$$(-\Delta)^{\frac{\alpha}{2}}u(x) = C_{d,\alpha}\left(\int_{y\in B_{r_0}(x)}\frac{u(x)-u(y)}{\|x-y\|_2^{d+\alpha}}dy + \int_{y\notin B_{r_0}(x)}\frac{u(x)-u(y)}{\|x-y\|_2^{d+\alpha}}dy\right). \tag{1.2}$$

It is shown that the fractional Laplacian of $u(x)$ can be calculated via the following approximation:

$$\begin{aligned}(-\Delta)^{\frac{\alpha}{2}}u(x) = {} & C_{d,\alpha}\frac{|S^{d-1}|r_0^{2-\alpha}}{2(2-\alpha)}\,\mathbb{E}_{\xi,r_I\sim f_I(r)}\left[\frac{2u(x)-u(x-r_I\xi)-u(x+r_I\xi)}{r_I^2}\right] \\ & + C_{d,\alpha}\frac{|S^{d-1}|r_0^{-\alpha}}{2\alpha}\,\mathbb{E}_{\xi,r_O\sim f_O(r)}\big[2u(x)-u(x-r_O\xi)-u(x+r_O\xi)\big],\end{aligned} \tag{1.3}$$

where $|S^{d-1}|$ denotes the surface area of S^{d-1}, ξ is uniformly distributed on the sphere S^{d-1}, and r_I, r_O can be quickly sampled via

$$\frac{r_I}{r_0}\sim \text{Beta}(2-\alpha,1), \quad \frac{r_0}{r_O}\sim\text{Beta}(\alpha,1). \tag{1.4}$$

More precisely, one can resort to the classic Monte Carlo sampling to handle the fractional Laplacian (see [12] for more details).

The main aim of this work is to extend the idea in [12] to more general nonlocal operators. Our new contributions are summarized as follows: