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Neural Networks with Local Converging Inputs (NNLCI) for Solving Conservation Laws, Part I: 1D Problems

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Abstract. A novel neural network method is developed for solving systems of conservation laws whose solutions may contain abrupt changes of state, including shock waves and contact discontinuities. In conventional approaches, a low-cost solution patch is usually used as the input to a neural network for predicting the high-fidelity solution patch. With that technique, however, there is no way to distinguish a smeared discontinuity from a smooth solution with large gradient in the input, and the two almost identical inputs correspond to two fundamentally different high-fidelity solution patches in training and predicting. To circumvent this difficulty, we use local patches of two low-cost numerical solutions of the conservation laws in a converging sequence as the input to a neural network. The neural network then makes a correct prediction by identifying whether the solution contains discontinuities or just smooth variations with large gradients, because the former becomes increasingly steep in a converging sequence in the input, and the latter does not. The inputs can be computed from low-cost numerical schemes with coarse resolution, in a local domain of dependence of a space-time location where the prediction is to be made. Despite smeared input solutions, the output provides sharp approximations of solutions containing shock waves and contact discontinuities. The method works effectively not only for regions with discontinuities, but also for smooth regions of the solution. It is efficient to implement, once trained, and has broader applications for different types of differential equations.

AMS subject classifications: 65L99, 65M99, 65N99, 68T99, 76L05

Key words: Neural network, neural networks with local converging inputs, physics informed machine learning, conservation laws, differential equation, multi-fidelity optimization.

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1 Introduction

There has been extensive research on numerical methods for solving conservation laws whose solutions may contain shock waves and contact discontinuities. Proposed methods of approaching the problem include the Godunov [14], MUSCL [45,46], ENO [16,40], and WENO [18,23] schemes, adaptive mesh refinement [4], moving mesh [42], hierarchical reconstruction [24,25,52], preconditioning schemes [17,54], space-time methods [50], and many others. These numerical techniques have been broadly implemented to study a wide variety of fluid dynamics and combustion problems [43,44,48,53]. The development of machine learning techniques for solving hyperbolic conservation laws, however, is still in an early stage.

In the past decade, data-driven modeling in machine learning has been developed for many scientific and engineering disciplines, including image processing, biomedical applications, and engineering design optimization [6, 7, 10, 20, 21, 29, 31]. Advances in computational resources, including improvements in graphics processing units (GPUs) and tensor processing units (TPUs), have accelerated training in deep learning frameworks such as TensorFlow and PyTorch for computer vision [51], natural language processing [47], and other applications [8].

Recently, Sirignano and Spiliopoulos [41] approximate the unknown solution as a mapping from a space-time location to the solution value there with a deep neural network (Deep Galerkin Method), incorporating the finite difference residue error and initial and boundary constraints in the loss function. E and Yu [12] incorporate the Ritz energy of a finite element method into the loss function of a neural network (Deep Ritz Method.) Raissi *et al.* [33] develop physics-informed neural networks (PINN) by employing an automatic differentiation [3] to define the residue error in the loss function. This method has achieved much success in data-driven methods for identifying nonlinear PDEs. It is capable of predicting fluid flow dynamics with given governing PDEs, such as the Navier-Stokes equations [34,35]. PINN has also been applied to solve several other physical problems. These include lattice Boltzmann equations with the Bhatnagar-Gross-Krook collision [26] and parabolic PDE, for heat transfer problems [5, 22], and so on. To explore the possibility that temporal evolution of learning could take place a priori in the absence of data, Psaros et al. [32] extended PINN to a meta-learning framework. The PINN algorithm usually is much more complex than conventional schemes for evolution equations, because in the loss function the equation is evaluated at space-time-filling points. It still remains a challenge for this algorithm to capture discontinuities and their ensuing interactions in the domain of interest.

In [9, 27], finite expansions of neural networks were introduced to form a mapping that can map the entire initial value and a spatial location to a high-fidelity solution at the same location at a later time. Earlier works [15, 38] had identified shock waves or contact surfaces from smeared solutions. In [28], the Rankine-Hugoniot jump conditions were added as a constraint to the loss function of the neural network for solving Riemann problems. In [11, 37], neural networks were used to detect discontinuities and