Analysis in Theory and Applications
Anal. Theory Appl., Vol. **39**, No. 3 (2023), pp. 287-298

DOI: 10.4208/ata.OA-2021-0051
September 2023

# Partial Derivatives, Singular Integrals and Sobolev Spaces in Dyadic Settings

Hugo Aimar[1], Juan Comesatti[1], Ivana Gómez[1,*] and Luis Nowak[2]

[1] *Instituto de Matemática Aplicada del Litoral "Dra. Eleonor Harboure", UNL, CONICET, CCT CONICET Santa Fe, Predio "Dr. Alberto Cassano", Colectora Ruta Nac. 168 km 0, Paraje El Pozo, S3007ABA Santa Fe, Argentina*
[2] *Instituto de Investigación en Tecnologías y Ciencias de la Ingeniería, Universidad Nacional del Comahue, CONICET; Departamento de Matemática-FaEA-UNComa, Neuquén, Argentina*

Received 20 October 2021; Accepted (in revised version) 23 June 2022

**Abstract.** In this note we show that the general theory of vector valued singular integral operators of Calderón-Zygmund defined on general metric measure spaces, can be applied to obtain Sobolev type regularity properties for solutions of the dyadic fractional Laplacian. In doing so, we define partial derivatives in terms of Haar multipliers and dyadic homogeneous singular integral operators.

**Key Words**: Sobolev regularity, Haar basis, space of homogeneous type, Calderón-Zygmund operator, dyadic analysis.

**AMS Subject Classifications**: 42C40, 26A33

## 1 Introduction

In order to properly state the underlying ideas in the main problems and results of this paper, let us start by a well known and classical result that we give in a form which is suitable for our setting. Set $\mathscr{S}(\mathbb{R}^n)$ to denote the class of Schwartz test functions of the theory of distributions in the Euclidean space $\mathbb{R}^n$. As usual we shall use $\widehat{f}$ to denote the Fourier transform of $f$. A linear operator $L : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$ is a second order differential operator in $\mathbb{R}^n$ of the form

$$L\varphi = \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2 \varphi}{\partial x_i \partial x_j}$$

*Corresponding author. *Email addresses:* `haimar@santafe-conicet.gov.ar` (H. Aimar), `jcomesatti@santafe-conicet.gov.ar` (J. Comesatti), `ivanagomez@santafe-conicet.gov.ar` (I. Gómez), `luis.nowak@faea.uncoma.edu.ar` (L. Nowak)

 

if and only if

$$\widehat{L\varphi}(\xi) = m(\xi)\widehat{\Delta\varphi}(\xi)$$

with $m(\xi)$ homogeneous of degree zero and $m(\xi)$ on the unit sphere $S^{n-1}$ of $\mathbb{R}^n$ is a second order polynomial. The proof is just the application of the basic rules for the Fourier Transform of derivatives. Hence, in a sense, the family of all constant coefficient second order differential operators can be identified with the class $\mathscr{M}$ of those special multipliers $m(\xi)$.

A lower order instance of the above remark is provided by first order derivatives defined in terms of the operator $\sqrt{-\Delta}$. In fact,

$$\widehat{\frac{\partial}{\partial x_j}} = c\frac{\xi_j}{|\xi|}(\sqrt{-\Delta})\widehat{}.$$

In other words, the partial derivatives can be regarded as the composition of $\sqrt{-\Delta}$ and a Calderón-Zygmund singular integral, actually the $j$-th Riesz transform whose Fourier multiplier is $\frac{\xi_j}{|\xi|}$. Aside from the relevance of this interplay between the Fourier Transform and differential operators in the theory of Calderón-Zygmund of Sobolev regularity, sometimes a Laplacian type operator and a particular Fourier type analysis are known, nevertheless there is no a natural way to understand partial or directional derivatives. The above identification of linear PDE with $\mathscr{M}$ provides an analytical natural way to define differential operators associated with the Laplacian of the setting. In this note we consider these problem in the dyadic setting. For simplicity we shall work our theory in $\mathbb{R}^+ = [0,\infty)$, since it becomes a quadrant for the standard dyadic intervals in $\mathbb{R}$. The theory extends naturally to more general underlying spaces and more general dyadic families. Our program includes the analysis of the homogeneity properties of kernels and multipliers, the definition of partial derivatives and gradient and the characterization through norms of the gradient of the Sobolev spaces provided by the dyadic fractional Laplacian. The main tool will be a vector valued theorem of singular integrals of Calderón-Zygmund type defined on a space of homogeneous type. Following ideas in [3] we give a model in an adequate space of homogeneous type of the dyadic setting, now with values in an infinite dimensional sequence space of the type $\ell^2$.

The paper is organized as follows. In Section 2 we describe the basic dyadic setting, including some partitions of the class of all dyadic intervals in subfamilies that will play a special role in the analysis of homogeneity properties of the kernels that we accomplish in Section 3. Section 4 is devoted to introduce directional and partial derivatives and to show that the $L^2$ norms of the gradient of $f$ is the right dyadic fractional energy contained in $f$. We also obtain the formulas, via singular integrals and the fractional dyadic Laplacian, for directional and partial derivatives and for the gradient in terms of a vector valued singular integral. In Section 5 we briefly sketch the vector valued theory of singular integrals on spaces of homogeneous type that we use in Section 6 in order to prove our main result showing the boundedness of the $L^p$ norm of the gradient of $f$ in terms of the $L^p$ norm of the dyadic fractional Laplacian $(-\Delta)^{s/2}_{dy}$.