East Asian Journal on Applied Mathematics
doi: 10.4208/eajam.2022-135.041222

Vol. **13**, No. 4, pp. 835-857
November 2023

Bounded Traveling Wave Solutions of a (2+1)-Dimensional Breaking Soliton Equation

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Received 24 April 2022; Accepted (in revised version) 4 December 2022.

Abstract. We qualitatively analyze the bounded traveling wave solutions of a (2+1)-dimensional generalized breaking soliton (gBS) equation by using the theory of planar dynamical systems. We present the global phase diagrams of the dynamical system corresponding to the (2+1)-dimensional gBS equation under different parameters. The conditions for the existence of bounded traveling wave solutions are successfully derived. We find the relationship between the waveform of bounded traveling wave solutions and the dissipation coefficient β. When the absolute value of the dissipation coefficient β is greater than a critical value, we find that the equation has a kink profile solitary wave solution, while the solution has oscillatory and damped property if $|\beta|$ is less than the critical value. In addition, we give the exact bell profile solitary wave solution and kink profile solitary wave solution by using undetermined coefficient method. The approximate oscillatory damped solution is given constructively. Through error analysis, we find that the approximate oscillatory damped solution is meaningful. Finally, we present the graphical analysis of the influence of dissipation coefficient β on oscillatory damped solution in order to better understand their dynamical behaviors.

AMS subject classifications: 35Q51, 35Q53, 35C99, 68W30, 74J35

Key words: Planar dynamical systems, bounded traveling wave solution, approximate oscillatory damped solution.

1. Introduction

Nonlinear evolution equations (NLEEs) are widely used to describe complex phenomena in various scientific fields, especially in physics. Exact traveling wave solutions of such equations are basic objects in mathematical physics, and many of them, including soliton solutions, breather solutions, lump solutions, peakon solutions, and Jacobi elliptic periodic solutions are deeply studied — cf. Refs. [1–3, 8, 12, 13, 20, 21, 23–27, 34]. There are also

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various effective methods to find exact traveling wave solutions of NLEEs, such as Hirota's bilinear method [9], Darboux transformation [18], Lie symmetry transformation [19], and variable separation method [17]. Besides, there is another method based on the theory of planar dynamical systems [14,16]. This method mainly uses the bifurcation theory of dynamic system, combined with the phase diagram of integrable traveling wave system, and obtains a large number of exact traveling wave solutions with the help of elliptic function and other tools.

In this paper, we make a qualitative analysis of a (2+1)-dimensional generalized breaking soliton (gBS) equation

$$u_t + \alpha u_{xxy} + 4\alpha u u_y + 4\alpha u_x \partial_x^{-1} u_y + \beta u_{xt} = 0, \tag{1.1}$$

where $\alpha > 0$ is a known constant and β is an arbitrary constant.

When $\beta = 0$, the Eq. (1.1) reduces to the following (2+1)-dimensional Bogoyovenskii's breaking soliton (BS) equation

$$u_t + \alpha u_{xxy} + 4\alpha u u_y + 4\alpha u_x \partial_x^{-1} u_y = 0, \tag{1.2}$$

its equivalent form is

$$\begin{aligned} &u_t + \alpha u_{xxy} + 4\alpha u u_y + 4\alpha u_x w = 0, \\ &u_y = w_x, \end{aligned}$$

which describes the (2+1)-dimensional interaction of a Riemann wave propagating along the y-axis with a long wave along the x-axis. The Eq. (1.2) was studied by Bogoyovenskii [4] and overlapping solutions were generated. Radha and Lakshmanan [22] showed that (1.2) possesses the Painlevé property and dromion like structures. A general variable separation solution of (1.2) is also derived by Zhang and Meng [30]. Fan and Hon [6] provide a detail asymptotic analysis procedure and a multidimensional generalization of cnoidal waves of (1.2). Exact breather-type and periodic-type soliton solutions, including double-breather-type soliton solutions, breather-type periodic soliton solutions, breather-type two-soliton solutions, periodic-type two-soliton, and three-soliton solutions of (1.2) are obtained using the extended three-wave method (ETM) [33]. Other BS equations with different parameters have been studied in [5,7,10,28,29].

If $\beta \neq 0$, the Eq. (1.1) has a dissipative effect. Since dissipation is inevitable in practical problems, Zhang studied some NLEEs with such an effect and found exact bounded traveling wave and approximate oscillatory damped solutions [11,15,31,32].

However, there are other important issues to be studied for the bounded traveling wave solutions of the Eq. (1.1), e.g. the influence of the dissipation coefficient β. And what conditions the dissipation coefficient β meets, there are oscillatory damped solutions. How to find exact or approximate oscillatory damped solutions, and if we get approximate oscillatory damped solutions, how to give the error estimate? This work will focus on these issues.

In this paper, we study bounded traveling wave solutions of the Eq. (1.1) by using the theory of planar dynamical systems. Under the influence of the dissipation coefficient β,