

Primal-Dual Active-Set Method for the Valuation Of American Exchange Options

Xin Wen¹, Haiming Song¹, Rui Zhang¹ and Yutian Li^{2,*}

¹*School of Mathematics, Jilin University, Changchun, Jilin, 130012, China.*

²*School of Science and Engineering, the Chinese University of Hong Kong, Shenzhen, Guangdong, 518172, China.*

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Abstract. An American exchange option is a rainbow option with two underlying assets, whose pricing model is a two-dimensional free boundary problem and is equivalent to a parabolic variational inequality problem on a two-dimensional unbounded domain. The present work proposes an effective numerical method for this complex problem. We first reduce the problem into a one-dimensional linear complementarity problem (LCP) on a bounded domain based on a dimension reduction transformation, an a priori estimate for the optimal exercise boundary, and a far-field truncation technique. This LCP is then approximated by a finite element method with a geometric partition in the spatial direction and a backward Euler method with a uniform partition in the temporal direction. The convergence order of the fully discretized scheme is established as well. Further, according to the features of the discretized system, a primal-dual active-set (PDAS) method is imposed to solve this problem to obtain the option price and the optimal exercise boundary simultaneously. Finally, several numerical simulations are carried out to verify the theoretical results and effectiveness of the proposed method.

AMS subject classifications: 35A35, 90A09, 65K10, 65M12, 65M60

Key words: American exchange option, linear complementarity problem, finite element method, primal-dual active-set method.

1. Introduction

The development of financial derivatives has received more and more attention in the wave of economic globalization in recent decades. The option, as an important financial derivative, is a contract that the holder could but is not obligated to purchase (call option) or sell (put option) a certain amount of the underlying asset at a prescribed price during a fixed time. The option price depends on the underlying asset's price and occupies a significant position in hedging strategy [18]. There are mainly two types of options, European and American ones. The former can only be exercised on the maturity date specified in the

*Corresponding author. *Email addresses:* songhaiming@jlu.edu.cn (H. Song), liyutian@cuhk.edu.cn (Y. Li)

contract, while the latter can be exercised on any business day on or before the maturity date. This leads to the distinction that the European option has a closed-form solution while the American option does not. Hence, numerical approximations are essential for American options.

The modern option pricing theory was founded by Black and Scholes' seminal work [3] in 1973. They introduced the Black-Scholes (B-S) equation for the option price, which could be used for various options with one risky asset. Since then, the valuation of options has attracted the attention of researchers. With the development of the financial market, the classic option defined on one risk asset can not satisfy the investor's demand [25]. Therefore, the option based on multiple risk assets, namely multi-asset options, is booming. In general, multi-asset options can be divided into three categories for portfolios: rainbow options [5], basket options [11], and quanto options [34]. An exchange option is a rainbow option that gives the option holder the right to exchange a given quantity of one asset for a given quantity of another. It is predominantly used in foreign exchange, fixed-income, and equity markets. Compared with the classic single-asset option, the exchange option has become one of the hot topics gradually in the field of option pricing because of its flexibility.

In 1978, Margrabe [24] was the first to establish the pricing formula of the European exchange option under the B-S framework, where correlated geometric Brownian motion processes modeled the underlying asset prices, and the payoff of the exchange option depended on the price difference between two underlying assets. In the same year, Fischer [10] also investigated the pricing issue of exchange options, and accounted for the situation where exercise price was the price of an untraded asset. The approach of Margrabe was generalized by Wong [32] in 2008, and they relaxed some restrictions on the drift terms of geometric Brownian motion processes. There are many models derived from the exchange options, such as spread options [23], power exchange options [33], and so on. This paper concentrated on the more significant American exchange option pricing problem.

Since the emergence of the B-S model, researchers have tried to derive explicit expression of option price for American options, but they all failed due to the presence of the optimal exercise boundary. There are now two approaches to solving the option pricing model, one is the analytical approximation, and the other is the numerical solution. Analytical approaches include the polynomial-time approximate algorithm [6], the Taylor expansion approximation [27], the time-recursive way [21], the fast Fourier transform [22], and others. For numerical methods for the valuation of American options, there are the binomial method [8], the Monte Carlo simulation [26], the finite difference [19] and the finite element method [28,29]. This paper studies the numerical method to price American exchange options.

According to the work of predecessors, there are three main difficulties in the numerical treatment of American exchange options:

- (1) The solution region is a two-dimensional unbounded domain, so it is not easy to design the numerical algorithm directly, and the calculation is time-consuming.
- (2) The pricing model is a backward variable-coefficient problem, and the optimal exercise boundary is unknown, dramatically increasing the problem's nonlinearity.