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Arbitrary High-Order Structure-Preserving Schemes for Generalized Rosenau-Type Equations

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Abstract. Arbitrary high-order numerical schemes conserving the momentum and energy of the generalized Rosenau-type equation are studied. Derivation of momentum-preserving schemes is made within the symplectic Runge-Kutta method coupled with the standard Fourier pseudo-spectral method in space. Combining quadratic auxiliary variable approach, symplectic Runge-Kutta method, and standard Fourier pseudo-spectral method, we introduce a class of high-order mass- and energy-preserving schemes for the Rosenau equation. Various numerical tests illustrate the performance of the proposed schemes.

AMS subject classifications: 65M06, 65M70

Key words: Momentum-preserving, energy-preserving, high-order, symplectic Runge-Kutta method, Rosenau equation.

1. Introduction

We consider the following generalized Rosenau-type equations:

$$\begin{aligned} &\partial_t u(x,t)+\kappa\partial_x u(x,t)-\delta\partial_{xxt}u(x,t)+b\partial_{xxx}u(x,t)\\ &\quad+\alpha\partial_{xxxxt}u(x,t)+\beta\partial_x\big(u(x,t)^p\big)=0, \qquad x\in\Omega\subset\mathbb{R},\quad t>0,\\ &u(x,0)=u_0(x), \qquad x\in\Omega\subset\mathbb{R}, \end{aligned} \tag{1.1}$$

where t and x are respectively time and spatial variables, $\kappa,\delta>0, b,\alpha>0$ and β given real constants, $u:=u(x,t)$ is a real-valued wave function, p a given positive integer, $u_0(x)$ an initial condition, and $\Omega=[x_l,x_r]$ a bounded domain. In what follows, the Rosenau equation (1.1) will be also supplemented by periodic boundary conditions.

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Initially, the Eq. (1.1) has been used to describe the dynamics of dense discrete systems [34]. Nowadays, it plays an important role in fluid mechanics of the atmosphere and ocean. Moreover, when u is assumed to be smooth, the Eq. (1.1) satisfies the following Hamiltonian formulation:

$$u_t = \mathscr{J}\frac{\delta \mathscr{H}}{\delta u}, \tag{1.2}$$

where $\mathscr{J} = -(1-\delta\partial_{xx}+\alpha\partial_{xxxx})^{-1}\partial_x$ is a Hamiltonian operator and $\mathscr{H}$ the Hamiltonian functional — i.e.

$$\mathscr{H}(t) = \int_\Omega \left(\frac{\kappa}{2}u^2 - \frac{b}{2}u_x^2 + \frac{\beta}{p+1}u^{p+1}\right)dx, \quad t \geq 0. \tag{1.3}$$

In addition to the Hamiltonian energy (1.3), the Eq. (1.1) also conserves the mass

$$\mathscr{M}(t) = \int_\Omega u dx \equiv \mathscr{M}(0), \quad t \geq 0, \tag{1.4}$$

and the momentum

$$\mathscr{I}(t) = \int_\Omega \left(\frac{1}{2}u^2 + \frac{\delta}{2}u_x^2 + \frac{\alpha}{2}u_{xx}^2\right)dx \equiv \mathscr{I}(0), \quad t \geq 0. \tag{1.5}$$

In order to solve such Hamiltonian partial differential equations, it is often preferable to use special numerical schemes, which would inherit one or more intrinsic properties of the original system exactly in a discrete sense. Note that such a kind schemes are called structure-preserving — cf. [13, 14, 21]. Chung [10] proposed an implicit finite difference (IFD) scheme, which can satisfy the discrete analogue of momentum (1.5) and proved that the scheme has the second-order accuracy both in time and space. Subsequently, Omrani *et al.* [30] developed a linearly implicit momentum-preserving finite difference scheme for the classical Rosenau equation, in which a linear system is to be solved at every time step. Thus, it is computationally much cheaper than that of the IFD scheme. Over the years, various momentum-preserving schemes for the Eq. (1.1) have been proposed and analyzed — cf. Refs [1, 2, 22, 28, 31, 42, 43, 45–47]. However, to the best of our knowledge, all of existing momentum-preserving schemes have at most second-order accuracy in time. It has been shown in [18, 26] that, compared with the second-order schemes, the high-order ones not only provide smaller numerical errors as a large time step chosen, but also are more advantages in the robustness. Consequently, one of the goals of this work is to present a novel paradigm for developing arbitrary high-order momentum-preserving schemes for the Eq. (1.1).

In addition to the momentum conservation law (1.5), the Eq. (1.1) satisfies the Hamiltonian energy (1.3), which is one of the most important first integrals of the Hamiltonian system. Cai *et al.* [3] proposed a second-order energy-preserving scheme based on the averaged vector field method [32] and two fourth-order energy-preserving schemes based on composition ideas [21]. Nevertheless, it is shown in [21] that the high-order schemes obtained by the composition method will be at the price of a terrible zig-zag of the step points — cf. [21, Fig. 4.2], which may be tedious and time consuming. Thus, the construction of high-order energy-preserving schemes for the Rosenau equation (1.1) seems to be still at