Numer. Math. Theor. Meth. Appl. Vol. **16**, No. 4, pp. 1087-1126
doi: 10.4208/nmtma.OA-2023-0082 November 2023

# On Lagrange-Projection Schemes for Shallow Water Flows Over Movable Bottom with Suspended and Bedload Transport

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Received 5 July 2023; Accepted (in revised version) 10 July 2023

**Abstract.** In the present work we aim to simulate shallow water flows over movable bottom with suspended and bedload transport. In order to numerically approximate such a system, we proceed step by step. We start by considering shallow water equations with non-constant density of the mixture water-sediment. Then, the Exner equation is included to take into account bedload sediment transport. Finally, source terms for friction, erosion and deposition processes are considered. Indeed, observe that the sediment particle could go in suspension into the water or being deposited on the bottom. For the numerical scheme, we rely on well-balanced Lagrange-projection methods. In particular, since sediment transport is generally a slow process, we aim to develop semi-implicit schemes in order to obtain fast simulations. The Lagrange-projection splitting is well-suited for such a purpose as it entails a decomposition of the (fast) acoustic waves and the (slow) material waves of the model. Hence, in subsonic regimes, an implicit approximation of the acoustic equations allows us to neglect the corresponding CFL condition and to obtain fast numerical schemes with large time step.

**AMS subject classifications**: 65M08, 76M12, 35L60
**Key words**: Shallow water flows, movable bottom, erosion-deposition processes, Lagrange-projection schemes, implicit-explicit schemes.

## 1. Introduction and mathematical model

Sediment transport is an interesting and active topic in the field of geophysical flows. Sediment is transported by the action of a river current or due to currents near

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coastal areas mainly in two ways: a suspended load (fine fractions carried by the flow) and bedload (coarse fractions which move close to the bottom rolling, jumping and sliding), see [54].

Knowledge of sediment transport has different practical applications. For instance, in civil engineering, to plan the extended life of a dam forming a reservoir. Moreover, sediment deposition downstream reduces river capacity in that area, which may be a potential problem in flood situations. Sediments also play an important role in some environmental problems as well. For example, suspended sediments have a direct impact on fish habitat in river or estuaries [39].

A first common approach to model sediment transport by a fluid is to couple the shallow water equations [2, 45, 47] with the so-called Exner equation [30]. Many works have been proposed to study such a problem, which depends on an empirical definition of the solid transport flux for bedload transport (see [10,11,29,40,46,50,52] among many others). This first approach is then completed by including some transport equations for suspended sediment, that is sediment particles which have been eroded from the bottom and remain floating in the current for some time until subsequent sedimentation are sedimented afterwards (see [25, 39, 41, 43, 44, 49, 51] among many others).

One of the key points of such problems is that the characteristic times associated to sediment transport dynamics is much larger than the one corresponding to fluids. Hence, studying sediment transport usually requires long time simulations to see sediment's evolution. As such, numerical simulations will run for long wall-clock times, which are carried out in small time steps, mainly dominated by the characteristic fluid speed.

To overcome this difficulty, different strategies have been proposed. The most common approach is to use semi-implicit schemes (see [13–16] among others). In particular, this approach is used in [7, 35], where bedload transport with the simple Grass formula is considered as well as variable density. Moreover, in [36], the authors propose a semi-implicit scheme based on the theta method for sediment bedload transport models with gravitational effects under subcritical regimes. Another approach is the use of the Lagrange-Projection strategy (see [12, 20–22, 28, 48] and references therein). This framework allows us to naturally decouple the acoustic terms of the model from the transport ones. Such a decomposition is useful and very efficient to deal with subsonic or near low-Froude number flows. In such cases, the usual CFL time step limitation of Godunov-type schemes is driven by the acoustic waves and can thus be very restrictive. The Lagrange-projection strategy allows us to design a very natural implicit-explicit and large time step scheme, with a CFL restriction based on the (slow) transport waves and not on the (fast) acoustic waves. Therefore, in this paper, we consider the Lagrange-projection technique adapted to the problem of sediment transport. In particular, we aim to define a semi-implicit scheme for sediment transport problems.

Hence, let us briefly present the corresponding mathematical model. It is deduced from the Navier-Stokes equations under the hypothesis that the horizontal scale is much greater than the vertical one, assuming hydrostatic pressure and incompressibility of