Journal of Computational Mathematics
Vol.42, No.1, 2024, 49–70.

http://www.global-sci.org/jcm
doi:10.4208/jcm.2201-m2021-0140

CONVERGENCE ANALYSIS OF SOME FINITE ELEMENT PARALLEL ALGORITHMS FOR THE STATIONARY INCOMPRESSIBLE MHD EQUATIONS*

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Abstract

By combination of iteration methods with the partition of unity method (PUM), some finite element parallel algorithms for the stationary incompressible magnetohydrodynamics (MHD) with different physical parameters are presented and analyzed. These algorithms are highly efficient. At first, a global solution is obtained on a coarse grid for all approaches by one of the iteration methods. By parallelized residual schemes, local corrected solutions are calculated on finer meshes with overlapping sub-domains. The subdomains can be achieved flexibly by a class of PUM. The proposed algorithm is proved to be uniformly stable and convergent. Finally, one numerical example is presented to confirm the theoretical findings.

Mathematics subject classification: 35Q30, 65M60, 65N30, 76D05.
Key words: Partition of unity method, Local and parallel algorithm, Finite element method, Iteration methods, Magnetohydrodynamics.

1. Introduction

The stationary incompressible MHD equations [1] in a Lipschitz polygon/polyhedron $\Omega \subset R^d$ $(d = 2, 3)$ with homogeneous Dirichlet boundary conditions are described as

$$-R_e^{-1}\Delta\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p - S_c\mathrm{curl}\,\mathbf{B}\times\mathbf{B} = \mathbf{f}, \tag{1.1}$$

$$\mathrm{div}\,\mathbf{u} = 0, \tag{1.2}$$

$$S_c R_{e_m}^{-1}\mathrm{curl}\,\mathrm{curl}\,\mathbf{B} - S_c\mathrm{curl}\,(\mathbf{u}\times\mathbf{B}) - \nabla r = \mathbf{g}, \tag{1.3}$$

$$\mathrm{div}\,\mathbf{B} = 0, \tag{1.4}$$

$$\mathbf{u}|_{\partial\Omega} = 0, \quad \mathbf{B}\times\mathbf{n}|_{\partial\Omega} = 0, \quad r|_{\partial\Omega} = 0, \tag{1.5}$$

where R_e and R_{e_m} are the Reynolds numbers of hydrodynamic and magnetic, respectively, $\mathbf{n}$ is the unit outward normal vector on $\partial\Omega$, S_c is the coupling number of the two fields. $\mathbf{u}$ represents

* Received May 13, 2021 / Revised version received October 8, 2021 / Accepted January 21, 2022 / Published online December 16, 2022 /
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fluid velocity field, $\mathbf{B}$ magnetic field strength, p hydrodynamic pressure and r magnetic pseudo-pressure. Let $\mathbf{g}$ be solenoidal.

The governing MHD model is strongly nonlinear because the classical equations of Maxwell and Navier-Stokes are coupled. This physical system describes the relationship between incompressible flows with electrically conducting property and the existing magnetic field. It has important applications in numerous areas of science, e.g., process metallurgy and MHD ion propulsion, see [2, 3].

Recently, finite element methods (FEM) for numerically solving MHD equations have become an attractive topic for the community of scientific computing. Based on the exact penalty constraint idea on magnetic, a stabilized FE formulation was studied in [4]. Stabilized FEM motivated by residual-based stabilizations was investigated in [5]. Divergence-cleaning algorithm in continuous FEM was given in [6]. A divergence-free discontinuous FEM was analyzed in [7]. To treat the nonlinear terms efficiently, three classical FE iterative methods were proposed and the stability and convergence related to physical parameters and iterations were proved by Dong *et al.* [8]. By using the Lagrange multiplier associated with the magnetic divergence constraint, a double-saddle-point FE formulation was given and analyzed in [9], and a mixed discontinuous Galerkin scheme of this version was proposed by Houston *et al.* [10]. The mixed FEMs with exactly preserving mass conservation of hydrodynamics and Gauss law of magnetic were studied in [11] and [12–15], respectively. Some robust solvers for finite element discrete system was designed in [16–18]. As for the time-dependent MHD equations, the stabilized nodal-based FEMs were proposed in [19], Euler semi-implicit fully discrete FE schemes were analyzed by Prohl [20] and He [21], the Crank-Nicolson extrapolation fully discrete FE scheme was analyzed by Dong and He [22].

It has been proven practically that two-level FEM [23, 24] is a high-efficiency technique to solve partial differential equations numerically, since it can reduce the cost of computing. This method has been applied to treat the nonlinear terms and coupled terms in the MHD problem in [25–27]. According to the observation of the behavior of a FE solution, [28] proposed parallel FEMs based on local algorithms. [28] obtained low frequencies component governing the global properties of the solution by using coarse mesh, and then approximates high frequencies one by solving the resulted local residual subproblems on several subdomains with the fine grids. This numerical algorithm is of high performance for few communications between blocks. Thus, it has been developed and extended to various problems, such as, Navier-Stokes equations [29–31], MHD equations [32, 33], etc.

Inspired by the algorithm in [28] and two-level FEM with respect to different physical parameters for the stationary MHD [27], in this article, we mainly extend the recent work [34, 35] to some local and parallel FE iterative algorithms (LPFEIAs) related to different physical parameters (Explicit in Theorem 3.2) for problem (1.1)-(1.5). The extensions to the previous studies [34, 35] are explained clearly before Theorem 3.12. According to different stable conditions of three classical m-iteration methods, we combine FEM with different iterations on a globally coarse mesh to obtain FE iterative solutions $(\mathbf{u}_H^m, \mathbf{B}_H^m, p_H^m, r_H^m)$ first, then we correct them by different linearized residual schemes in parallel on some local overlapping subdomains Ω_j to seek the correction solutions $(\mathbf{u}_{mh}^j, \mathbf{B}_{mh}^j, p_{mh}^j, r_{mh}^j), j = 1, \ldots, J$, where J is the number of the subdomains, m is the iterative step and mesh sizes satisfy h ($h \ll H$). Moreover, the uniform stability and convergence of each algorithm is analyzed.

The paper is divided into 4 sections. The next section is devoted to giving some notation preparation and providing some results of FEM for the problem (1.1)-(1.5). In Section 3, some