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Group-Invariant Solutions and Conservation Laws of One-Dimensional Nonlinear Wave Equation

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Abstract Based on classical Lie symmetry method, the one-dimensional nonlinear wave equation is investigated. By using four-dimensional subalgebras of the equation, the invariant groups and commutator table are constructed. Furthermore, optimal system of the equation is obtained, and the exact solutions can be gained by solving reduced equations. Finally, a complete derivation of the conservation law is given by using conservation multipliers.

Keywords One-dimensional nonlinear wave equation, Lie symmetry, optimal system, conservation law

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1. Introduction

Wave equations describe various wave phenomena and have a wide range of applications in the fields of physics [19], biology and engineering [22,25], making the solution of wave equations indispensable. The methods of solving partial differential equations (PDEs) mainly include (G'/G) expansion method [6,14,35], extended hyperbolic method [26], inverse scattering method [7], exponential function method [32], generalized exp-function method [13], Bäcklund transformation method [33], Jacobi elliptic method [2,4,15], hyperbolic tangent method [1], F-expansion method [11], homogeneous equilibrium method [24], Lie symmetry analysis method [12,20,21,29] and so on [3,9,10,34].

The Lie symmetry method can solve PDEs efficiently. In this article, we consider a one-dimensional nonlinear wave equation

$$u_{tt} = \Big((1+u)^{2a} u_x\Big)_x, \tag{1.1}$$

where u is a function of x, t and $a > 0$ is a constant. In [5], Ames, Lohner and Adams proposed a general nonlinear fluctuation equation

$$u_{tt} = [\mathscr{B}(u)u_x]_x, \tag{1.2}$$

where $\mathscr{B}$ is expressed as a function of u. Then, they discussed equation (1.2) with Lie symmetric analysis, and derived explicit invariant solutions to wave propagation and

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transonic equation in gases. Furthermore, Sophocleous and Kingston [27] attempted the following three special cases

$$u_{tt} = F\left(u_x\right) u_{xx}, \quad u_{tt} = F\left(u_{xx}\right), \quad u_{xt} = F\left(u\right). \tag{1.3}$$

Equation (1.3) exists in the discrete symmetries groups which form finite order cyclic groups. In [16], Hu studied the degenerate initial-boundary value problem of equation (1.1), and obtained the global existence of the solution by using the eigendecomposition method under relaxed conditions. The global existence of the solution to a more general 2×2 conservation system of equation (1.1) was proven in [18, 30]. In [8, 31], the conservation system of equation (1.1) has been studied. In [28], Sugiyama introduced the large-time behavior of the solution of the equation under the Cauchy problem, obtained the sufficient conditions for the degradation of the equation in finite time and derived a threshold for the global existence and degradation of the separated solution.

This article mainly includes the following sections. In the second section, the concepts of Lie symmetry and prolongation method are introduced, followed by a study on Lie point symmetry and one-dimensional optimal system of the equation. Investigating the group invariant solutions of the equation by the optimal system, we obtain the exact solutions through symmetry reduction. The third section discusses the conservation laws of the equation. Finally, a simple summary is drawn.

2. Lie symmetry analysis and optimal system of equation (1.1)

2.1. Definition introduction

Based on the conclusions of Sophus Lie, some concepts of Lie symmetry [23] have been set up.

Assume that the s-order partial differential equation system Q with q independent variable and m dependent variable is

$$\Delta q\left(x, u^{(n)}\right) = 0\ , \quad Q = 1, 2, 3, \cdots, k, \tag{2.1}$$

in which $x = \left(x^1, x^2, \cdots, x^q\right)$, $u = \left(u^1, u^2, \cdots, u^m\right)$ and $u^{(n)}$ represents arbitrary order derivative of u, and its range of value is from 0 to n. Now, let us discuss the infinitesimal one-parameter Lie group transformation of the system

$$\bar{x}^k = x^k + \varepsilon \xi^k\left(x, u\right) + o\left(\varepsilon^2\right), \quad \bar{u}^p = u^p + \varepsilon \phi^p\left(x, u\right) + o\left(\varepsilon^2\right), \tag{2.2}$$

where ε is an arbitrary, and ξ^k, ϕ^p represent the infinitesimal transformations of function independent variables and dependent variables respectively.

Considering the n-order differential equations for u

$$\Delta\left(x, u^{(n)}\right) = 0, \tag{2.3}$$

in which Δ denotes a smooth mapping from $X \times U^{(n)}$ to $\mathbb{R}$:

$$\Delta : X \times U^{(n)} \to \mathbb{R}.$$