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Interval Oscillation Criteria for Nonlinear Neutral Impulsive Differential Equations*

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Abstract In this paper, we study the interval oscillation for nonlinear neutral impulsive differential equations. Sufficient condition for the interval oscillation of the equations is obtained by using Riccati transformation and estimating the ratio of unknown functions $y(t-\sigma(t))$ and $y(t)$. Some known results are generalized and improved. An example is given to illustrate the results.

Keywords Interval oscillation theory, neutral differential equation, impulsive

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1. Introduction

The theory of impulsive differential equations has been extensively investigated and developed due to their potential applications in many fields such as bursting rhythm models in medicine and biology, optimal control models in economics, pharmacokinetics and frequency modulation systems, etc. The monographs [4, 6, 7, 10] are the applications of impulsive differential equations.

In 2007, Liu and Xu [8] gave interval oscillation criteria for equations of the form

$$\begin{cases} (r(t)x'(t))' + q(t)\Phi_\gamma(x(t)) = f(t), \quad t \neq \tau_i, \\ x(\tau_k^+) = a_k x(\tau_k), \quad x'(\tau_k^+) = b_k x'(\tau_k), \quad k = 1, 2, ... \end{cases}$$

They studied the interval oscillation of impulsive differential equations without delay parameters.

In 2010, Huang and Feng [3] considered the following second-order nonlinear impulsive differential equations with constant delay

$$\begin{cases} x''(t) + q(t)g(x(t-\sigma)) = f(t), \quad t \neq \tau_i, \\ x(\tau_k^+) = a_k x(\tau_k), \quad x'(\tau_k^+) = b_k x'(\tau_k), \quad k = 1, 2, ... \end{cases}$$

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In 2016, Zhou and Wang [16] studied the second-order nonlinear impulsive differential equations with variable delay of the form

$$\begin{cases} (r(t)x'(t))' + q(t)g(x(t-\sigma(t))) = f(t), & t \neq \tau_i, \\ x(\tau_k^+) = a_k x(\tau_k), \quad x'(\tau_k^+) = b_k x'(\tau_k), & k = 1, 2, ..., \end{cases}$$

and obtained some results which developed some known results of [3] and [13].

For the impulsive equations, almost all of the interval oscillation results in the existing literature were established for the case of "without delay" or "with constant delay" or "with variable delay" in a non-neutral case (see [2,5,12,15,17]). However, for the case of "with neutral", there are only oscillation results (see [1,9,11,14]), and the research on interval oscillation of impulsive differential equations is very scarce.

Motivated by the above papers, we consider the interval oscillatory behavior of solutions to the following neutral impulsive differential equation

$$\begin{gathered} (r(t)(x(t) + p(t)x(t-\sigma(t)))')' + q(t)g(x(t-\sigma(t))) = f(t), \quad t \geqslant t_0, \quad t \neq \tau_k, \\ x(\tau_k^+) = a_k x(\tau_k), \quad x'(\tau_k^+) = b_k x'(\tau_k), \quad k = 1, 2, ..., \end{gathered} \tag{1.1}$$

where $\{\tau_k\}$, $k = 1, 2, ...$ denotes the impulse moment sequence, $0 \leqslant t_0 = \tau_0 < \tau_1 < \tau_2 < ... < \tau_k < ...$ and $\lim\limits_{k\to\infty} \tau_k = \infty$, $x(\tau_k^+) = \lim\limits_{\varepsilon\to 0^+} x(\tau_k + \varepsilon)$ and $x(\tau_k^-) = \lim\limits_{\varepsilon\to 0^-} x(\tau_k + \varepsilon)$ represent the right and left limits of $x(t)$ at $t = \tau_k$. Denote $\pi(a, b) := \int_a^b \frac{1}{r(s)} ds$, and $y(t) = x(t) + p(t)x(t - \sigma(t))$.

By a solution to equation (1.1), we mean that a function $x(t)$ is piecewise continuous on the interval (t_0, ∞) with discontinuities of the first kind only at $t = \tau_k$, $k = 1, 2, ...$, i.e., at the moments of impulse the following relation is satisfied:

$$x(\tau_k^+) = a_k x(\tau_k), \quad x'(\tau_k^+) = b_k x'(\tau_k), \quad k = 1, 2, ...$$

As is customary, a nontrivial solution $x(t)$ of (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative. Otherwise, it is said to be nonoscillatory. In other words, a solution is said to be oscillatory if there exists an increasing divergent sequence $\{\xi_k\}_{k\in\mathbb{N}} \subset [t_0, \infty)$ such that $x(\xi_k^+)x(\xi_k^-) \leqslant 0$ for all $k \in \mathbb{N}$.

This paper is structured as follows. In Section 2, we present necessary notations, lemmas and definitions. In Section 3, we state and prove our main results. At last, one illustrative example is proposed.

2. Preliminaries

In this section, we will present some necessary knowledge and notations.

Let $I \subset \mathbb{R}$ be an interval. A function set $PLC(I, \mathbb{R})$ is defined as follows:

$PLC(I, \mathbb{R}) := \{z : I \to \mathbb{R} \mid z$ is continuous on $I/\{\tau_i\}$ and at each τ_i, $z(\tau_i^+)$ and $z(\tau_i^-)$ exist, and the left continuity of z is assumed, i.e., $z(\tau_i^-) = z(\tau_i)$, $i \in \mathbb{N}\}$.

We introduce a function set $\Omega(c, d)$ as follows:

$$\Omega(c, d) := \{w \in C^1[c, d] : w(t) \not\equiv 0, w(c) = w(d) = 0\}.$$

Throughout this paper, we assume that the following hypotheses are satisfied: