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Omni-Representations of Leibniz Algebras

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Abstract. In this paper, first we introduce the notion of an omni-representation of a Leibniz algebra $\mathfrak{g}$ on a vector space V as a Leibniz algebra homomorphism from $\mathfrak{g}$ to the omni-Lie algebra $\mathfrak{gl}(V)\oplus V$. Then we introduce the omni-cohomology theory associated to omni-representations and establish the relation between omni-cohomology groups and Loday-Pirashvili cohomology groups.

AMS subject classifications: 17B99, 17B10

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1 Introduction

Leibniz algebras were first discovered by Bloh [5] who called them D-algebras. Then Loday [18] rediscovered this algebraic structure and called them Leibniz algebras. A Leibniz algebra is a vector space $\mathfrak{g}$, endowed with a linear map $[\cdot,\cdot]_\mathfrak{g}: \mathfrak{g}\otimes\mathfrak{g}\longrightarrow\mathfrak{g}$ satisfying

$$\big[x,[y,z]_\mathfrak{g}\big]_\mathfrak{g}=\big[[x,y]_\mathfrak{g},z\big]_\mathfrak{g}+\big[y,[x,z]_\mathfrak{g}\big]_\mathfrak{g}, \quad \forall x,y,z\in\mathfrak{g}. \tag{1.1}$$

In particular, if the bracket $[\cdot,\cdot]_\mathfrak{g}$ is skew-symmetric, then it is a Lie algebra. Leibniz algebras have important applications in both mathematics and mathematical physics, e.g. the section space of a Courant algebroid is a Leibniz algebra [19]

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and the underlying algebraic structure of an embedding tensor is also a Leibniz algebra which further leads to applications in higher gauge theories [15].

The theory of representations of Leibniz algebras was introduced and studied in [18].

Definition 1.1. *A representation of a Leibniz algebra $(\mathfrak{g},[\cdot,\cdot]_{\mathfrak{g}})$ is a triple (V,l,r), where V is a vector space equipped with two linear maps $l:\mathfrak{g}\longrightarrow\mathfrak{gl}(V)$ and $r:\mathfrak{g}\longrightarrow\mathfrak{gl}(V)$ such that the following equalities hold:*

$$l_{[x,y]_{\mathfrak{g}}}=[l_x,l_y],\quad r_{[x,y]_{\mathfrak{g}}}=[l_x,r_y],\quad r_y\circ l_x=-r_y\circ r_x,\quad \forall x,y\in\mathfrak{g}. \tag{1.2}$$

Especially, faithful representations of Leibniz algebras were studied by Barnes [3], conformal representations of Leibniz algebras were studied by Kolesnikov [13], dual representations of Leibniz algebras were given in [21] in their study of Leibniz bialgebras. Representations of symmetric Leibniz algebras were studied by Benayadi [4].

Note that a representation of a Lie algebra $\mathfrak{g}$ on a vector space V is a Lie algebra homomorphism from $\mathfrak{g}$ to the Lie algebra $\mathfrak{gl}(V)$, which realizes an abstract Lie algebra as a subalgebra of the general linear Lie algebra. While the above representation of a Leibniz algebra does not have this advantage. The purpose of this paper is to introduce a new representation theory so that it can realize an abstract Leibniz algebra as a subalgebra of a concrete Leibniz algebra. The omni-Lie algebra $\mathfrak{gl}(V)\oplus V$ introduced by Weinstein [22] is naturally a Leibniz algebra and the main ingredient in our study. We introduce the notion of an omni-representation of a Leibniz algebra $\mathfrak{g}$ on a vector space V which is a homomorphism from $\mathfrak{g}$ to the Leibniz algebra $\mathfrak{gl}(V)\oplus V$. We show that a usual representation (V,l,r) gives rise to an omni-representation $\rho=(l^*\otimes 1+1\otimes l)+r$ of $\mathfrak{g}$ on $V^*\otimes V$.

The cohomology theory of Leibniz algebras was also developed by Loday and Pirashvili [18]. See [1, 6–9, 11] for more applications of Loday-Pirashvili cohomologies. We also develop the omni-cohomology theory for omni-representations introduced above, and give the relation between omni-cohomology groups and Loday-Pirashvili cohomology groups.

The paper is organized as follows. In Section 2, we restudy representation of Leibniz algebras and the corresponding semidirect products. In Section 3, we introduce the notion of omni-representations of Leibniz algebras and study the relation between omni-representations and the usual representations. In Section 4, we introduce omni-cohomology groups for Leibniz algebras with coefficients in omni-representations, and establish the relation between omni-cohomology groups and Loday-Pirashvili cohomology groups.