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CONVERGENCE OF THE FINITE VOLUME METHOD FOR STOCHASTIC HYPERBOLIC SCALAR CONSERVATION LAWS: A PROOF BY TRUNCATION ON THE SAMPLE-TIME SPACE

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Abstract. We prove the almost sure convergence of the explicit-in-time Finite Volume Method with monotone fluxes towards the unique solution of the scalar hyperbolic balance law with locally Lipschitz continuous flux and additive noise driven by a cylindrical Wiener process. We use the standard CFL condition and a martingale exponential inequality on sets whose probabilities are converging towards one. Then, with the help of stopping times on those sets, we apply theorems of convergence for approximate kinetic solutions of balance laws with stochastic forcing.

Key words. Finite volume method, stochastic balance law, kinetic formulation.

Contents

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1. Introduction

Stochastic hyperbolic scalar balance law. Let $T > 0$ be a finite time and $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t), (\beta_k(t)))_{t\in[0,T]}$ be a stochastic basis. Consider the hyperbolic scalar balance law with stochastic forcing

$$du(x,t) + \operatorname{div}_x(A(u(x,t)))dt = \Phi dW(t), \quad x \in \mathbb{T}^N, t \in (0,T). \tag{1}$$

Equation (1) is periodic in the space variable $x \in \mathbb{T}^N$, where $\mathbb{T}^N$ is the N-dimensional torus.

Assumption 1.1. The flux function A in (1) is supposed to be of class C^2: $A \in C^2(\mathbb{R};\mathbb{R}^N)$. We assume that A and its derivatives have at most polynomial growth. We denote its first derivative $A^{'} =: a$. Without loss of generality, we assume that $A(0) = 0$.

Assumption 1.2. The right-hand side of (1) is a stochastic increment in infinite dimension. It is defined as follows (see [8] for the general theory): $t \in [0,T] \mapsto W(t)$ is a cylindrical Wiener process, that is $\forall t \in [0,T], W(t) = \sum_{k\geq 1} \beta_k(t) e_k$, where the coefficients β_k are independent standard Brownian motions and $(e_k)_{k\geq 1}$ is an orthonormal basis of the separable Hilbert space H. Denoting $L_2(H,\mathbb{R})$ the set of Hilbert-Schmidt operators from H to the space of real numbers $\mathbb{R}$, we assume that

$$\Phi \in L_2(H,\mathbb{R}). \tag{2}$$

The Cauchy Problems. Let us quote the main results: In [14], E, Khanin, Mazin, Sinai proved uniqueness and existence of the solution of the stochastic Burgers Equation with additive noise carried by a cylindrical Wiener process. They used a periodic solution in space dimension one $x \in \mathbb{T}$ in order to prove the existence of an invariant measure. In [22], Kim proved uniqueness and existence of the solution for a more general non-linear flux, with the space variable $x \in \mathbb{R}$ and a real Brownian motion. In [16], Feng and Nualart proved uniqueness of a solution in space dimension $N \in \mathbb{N}^*$, while existence was proved only in space dimension one. They used for the first time a multiplicative noise. The existence of a solution in space dimension $N \in \mathbb{N}^*$ was proved later by Chen, Ding, Karlsen in [7]. In [5], Bauzet, Vallet and Wittbold proved uniqueness and existence of the solution for a non-linear flux, with the space variable $x \in \mathbb{R}^N$ and a multiplicative noise driven by a real Brownian motion. In [9], Debussche and Vovelle proved uniqueness and existence of the solution for a non-linear flux and a multiplicative noise driven by a cylindrical Wiener process. Their solution is periodic in space: $x \in \mathbb{T}^N$. All the previous solutions are entropic solutions defined by Kruzkov in [25]. While similar, slight differences on assumptions or formulations always exist. For exemple [5] is following the formulation of Di Perna [10] with the measure-valued solution, while [9] is following the kinetic formulation of Lions, Perthame, Tadmor [27]. They all first proved uniqueness, then existence via the approximation given by the stochastic parabolic equation. In [12], we followed the works of [9] by using a kinetic formulation, a multiplicative noise driven by a cylindrical Wiener process, and defining a periodic solution for the space variable $x \in \mathbb{T}^N$. We proved uniqueness of a solution, and a general framework for convergence of approximate solutions towards the unique solution.