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Numerical Computation of Helical Waves in a Finite Circular Cylinder using Chebyshev Spectral Method

Xing-Liang Lyu and Wei-Dong Su*

State Key Laboratory for Turbulence and Complex Systems and Department of Mechanics and Engineering Science, College of Engineering, Peking University, Beijing 100871, China

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Abstract. Helical waves are eigenfunctions of the curl operator and can be used to decompose an arbitrary three-dimensional vector field orthogonally. In turbulence study, high accuracy for helical waves especially of high wavenumber is required. Due to the difficulty in analytical formulation, the more feasible strategy to obtain helical waves is numerical computation. For circular cylinders of finite length, a semi-analytical method via infinite series to formulate the helical wave is known [E. C. Morse, J. Math. Phys., 46 (2005), 113511], where the eigenvalues are evaluated by iterating transcend equations. In this paper, the numerical computation for helical wave in a finite circular cylinder is implemented using a Chebyshev spectral method. The solving is transformed into a standard matrix eigenvalue problem. The large eigenvalues are computed with high precision, and the calculation cost to rule out spurious eigenvalues is significantly reduced with a new criterion suggested.

AMS subject classifications: 65N25, 65N35, 76F02, 76W05

Key words: Helical wave, Chandrasekhar-Kendall mode, eigenvalue problem, Chebyshev spectral method, spurious eigenvalues, circular cylinder.

1 Introduction

1.1 Background

Helical waves are the eigenfunctions of the curl operator, which was first noted by Beltrami [1]. It is also called force-free fields [2] in physics, or Taylor state [3] associating with a variational principle, and helical Fourier modes [4,5]. The terminology of "helical wave" has been introduced by Moffatt [6]. They have been widely used in fluid dynamics and magnetohydrodynamics. Yoshida and Giga [7] proved that helical waves can span a

*Corresponding author.
Email: swd@pku.edu.cn (W.-D. Su)

complete Hilbert space for solenoidal vector fields under proper boundary conditions in bounded domains of three-dimensional (3-D) space, and the corresponding helical wave decomposition (HWD) has been employed to analyze the global energy spectrum of turbulence in several cases [8]. In particular, the spectral coefficients of helical waves are scalars, and HWD can be taken as the generalization of Fourier basis functions in general bounded domains.

HWD plays an essential role in studying the energy cascade of homogeneous turbulence. Waleffe [9] investigated the nonlinear energy transfer of homogeneous turbulence and found that the scale and polarities of helical modes affect the direction of energy transfer. Chen et al [10] proved that the nonlinear transfer between opposite polarities permits the joint cascade of energy and helicity. Biferale et al [11] showed that an inverse energy cascade also occurs in 3-D isotropic turbulence by keeping only triadic interactions between sign-defined helical modes. And they also investigated the transfer properties of energy and helicity fluctuations by keeping only those triads that have sign-definite helicity content [12]. Helical waves are also used to study the decoupling mechanism between two-dimensional (2-D) modes and 3-D modes in rotating homogeneous turbulence [13, 14].

For magnetic fields, helical modes play an essential role in the coupling mechanism between the external helicity source and the driven plasma [15, 16]. Due to its completeness as functional basis, HWD are also be used for hydrodynamics simulations using Galerkin spectral representation [17–21], or helicity calculation [22]. In addition, helical waves can also be used to construct vortex knots [23]. An early standard reference to introduce the interdisciplinary application of helical waves comes from Moses [24].

Helical waves in an arbitrary bounded and simply connected domain $\mathcal{D}$ are solutions of the eigenfunction equation

$$\nabla\times\boldsymbol{B}=k\boldsymbol{B} \tag{1.1}$$

with the homogeneous boundary condition

$$\boldsymbol{B}\cdot\boldsymbol{n}|_{\partial\mathcal{D}}=0, \tag{1.2}$$

where $\boldsymbol{B}$ denotes helical wave and k denotes eigenvalue. As proved by Yoshida and Giga [7], an arbitrary solenoidal field in 3-D space can be decomposed into a series of orthogonal helical waves. This is the basis for the multi-scale expansion of vector fields, which can naturally be applied to turbulence studies.

Taking the curl of Eq. (1.1) again yields

$$\nabla^2\boldsymbol{B}+k^2\boldsymbol{B}=\boldsymbol{0}, \tag{1.3}$$

which is the Helmholtz equation that looks more familiar. However, Eq. (1.3) is just a necessary but insufficient condition for Eq. (1.1) to be satisfied. The case of $k=0$ is the well-known potential field, thus only nonzero eigenvalue k is concerned in the study.