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Error Analysis of the Nonconforming P_1 Finite Element Method to the Sequential Regularization Formulation for Unsteady Navier-Stokes Equations

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Abstract. In this paper we investigate the nonconforming P_1 finite element approximation to the sequential regularization method for unsteady Navier-Stokes equations. We provide error estimates for a full discretization scheme. Typically, conforming P_1 finite element methods lead to error bounds that depend inversely on the penalty parameter ϵ. We obtain an ϵ-uniform error bound by utilizing the nonconforming P_1 finite element method in this paper. Numerical examples are given to verify theoretical results.

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1 Introduction

Let Ω be a bounded convex polygon domain of $\mathbb{R}^2$ or $\mathbb{R}^3$ and Γ its boundary. We consider the unsteady Navier-Stokes equations for a viscous incompressible fluid in $\Omega\times[0,T]$:

$$\boldsymbol{u}_t-\nu\Delta\boldsymbol{u}+\boldsymbol{u}\cdot\nabla\boldsymbol{u}+\nabla p=\boldsymbol{f}_{ext}, \tag{1.1a}$$

$$\mathrm{div}\boldsymbol{u}=0, \tag{1.1b}$$

$$\boldsymbol{u}|_\Gamma=\boldsymbol{0}, \quad \boldsymbol{u}(\boldsymbol{x},0)=\boldsymbol{u}_0(\boldsymbol{x}). \tag{1.1c}$$

Here $\boldsymbol{u}(\boldsymbol{x},t)$ is the velocity of the fluid, p the pressure acting on the fluid, $\boldsymbol{f}_{ext}$ the external force, $\boldsymbol{u}_0$ the initial velocity and ν the dynamic viscosity. The Eqs. (1.1a)-(1.1c) can be written as the equivalent system below:

$$\begin{aligned}&\boldsymbol{u}_t-\nu\Delta\boldsymbol{u}+B(\boldsymbol{u},\boldsymbol{u})+\nabla p=\boldsymbol{f}_{ext},\\&\mathrm{div}\boldsymbol{u}=0,\\&\boldsymbol{u}|_\Gamma=\boldsymbol{0}, \quad \boldsymbol{u}(\boldsymbol{x},0)=\boldsymbol{u}_0(\boldsymbol{x}),\end{aligned}$$

where

$$B(\boldsymbol{u},\boldsymbol{u})=\boldsymbol{u}\cdot\nabla\boldsymbol{u}+\frac{1}{2}(\mathrm{div}\boldsymbol{u})\boldsymbol{u}.$$

Eqs. (1.1a)-(1.1c) present a long-recognized difficulty for numerical solution due to the coupling of $\boldsymbol{u}$ and p by the incompressible equation, where the pressure p does not explicitly appear. This results in an index-2 differential algebraic system (cf. [5, 11]) and may cause temporal instability in maintaining the algebraic constraint (or the incompressible equation in the Navier-Stokes context). Hence, direct discretization is not recommended. To overcome this difficulty, several methods have been proposed, such as the projection method (cf. [8, 15]), penalty method (cf. [4, 14]), iterative penalty method for steady problems (cf. [7]), Baumgarte stabilization (cf. [3]), and sequential regularization method (SRM) [11]. The SRM is based on methods for solving differential algebraic equations (cf. [1, 2]) and can be understood as a combination of the penalty method and Baumgarte stabilization (see [13]). It reads as follows: given $p_0(\boldsymbol{x},t)$ the initial guess, for $s=1,2,\ldots$, solve

$$(\boldsymbol{u}_s)_t-\nu\Delta\boldsymbol{u}_s+B(\boldsymbol{u}_s,\boldsymbol{u}_s)+\nabla p_s=\boldsymbol{f}_{ext}, \tag{1.2a}$$

$$\mathrm{div}(\alpha_1(\boldsymbol{u}_s)_t+\alpha_2\boldsymbol{u}_s)=\epsilon(p_{s-1}-p_s), \tag{1.2b}$$

$$\boldsymbol{u}_s|_\Gamma=\boldsymbol{0}, \quad \boldsymbol{u}_s(\boldsymbol{x},0)=\boldsymbol{u}_0(\boldsymbol{x}), \tag{1.2c}$$

where α_1 and α_2 are nonnegative constants and ϵ a small penalty parameter. It has been showed that $u-u_s$ and $p-p_s=\mathcal{O}(\epsilon^s)$. In other words, unlike the penalty