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Existence and Uniqueness for the Non-Compact Yamabe Problem of Negative Curvature Type

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Abstract. We study existence and uniqueness results for the Yamabe problem on non-compact manifolds of negative curvature type. Our first existence and uniqueness result concerns those such manifolds which are asymptotically locally hyperbolic. In this context, our result requires only a partial C^2 decay of the metric, namely the full decay of the metric in C^1 and the decay of the scalar curvature. In particular, no decay of the Ricci curvature is assumed. In our second result we establish that a local volume ratio condition, when combined with negativity of the scalar curvature at infinity, is sufficient for existence of a solution. Our volume ratio condition appears tight. This paper is based on the DPhil thesis of the first author.

Key Words: Yamabe problem, non-compact manifolds, negative curvature, asymptotically locally hyperbolic, asymptotically warped product, relative volume comparison, non-smooth conformal compactification.

AMS Subject Classifications: 35J91, 53C18, 53C25, 58J32, 58J90

1 Introduction

We are interested in the Yamabe problem on non-compact manifolds: given some complete non-compact Riemannian manifold (M,g) of dimension $n \geq 3$, does there exist a corresponding complete conformal metric whose scalar curvature is constant? Equivalently, we would like to find a complete metric $\tilde{g} = u^{\frac{4}{n-2}}g$, where u is some strictly positive smooth function on M solving the Yamabe equation

$$S_{\tilde{g}} = u^{-\frac{n+2}{n-2}}\left(-c_n\Delta_g u + S_g u\right) \equiv \text{constant}, \qquad c_n := \frac{4(n-1)}{n-2}.$$

Here, S_g and $S_{\tilde{g}}$ refer to the scalar curvatures of the corresponding metrics. The operator $-c_n\Delta_g + S_g$ is known as the conformal Laplacian.

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In the case that M is compact, the Yamabe problem has been extensively studied. The existence of a solution was established through the combined works of Yamabe [49], Trudinger [45], Aubin [7] and Schoen [43]. For other aspects of the Yamabe problem, see e.g., [10, 11, 13, 15, 20, 21, 31, 32, 35–38, 48] and references therein.

Our work is focused on the Yamabe problem of "negative curvature type" on non-compact manifolds, namely we are interested in obtaining conformal changes to constant negative scalar curvature. Consequently, asserting that $S_{\tilde{g}} \equiv -n(n-1)$, the equation for the conformal change in scalar curvature yields the Yamabe equation

$$-c_n \Delta_g u + S_g u = -n(n-1) u^{\frac{n+2}{n-2}}. \tag{Ya}$$

Finding a solution to the Yamabe problem thus amounts to finding a positive solution of the equation (Ya) above for which the corresponding conformal metric is complete.

The Yamabe problem of negative curvature type on non-compact manifolds has been studied extensively in the literature. Important progress has been made by Andersson, Chruściel and Friedrich [1], Gover and Waldron [24], Graham [22] and Loewner and Nirenberg [34]. For further literature, see e.g., [3–5,8,16–18,25,27–29,33,34,39,40,42,44,46] and references therein.

In the present work we consider conditions for existence of a solution to the Yamabe problem on a given non-compact Riemannian manifold (M, g), where g has asymptotically negative scalar curvature. We assume throughout the paper that g is complete and satisfies a condition of the type

$$\limsup S_g \leq -\varepsilon < 0 \tag{1.1}$$

for some $\varepsilon > 0$ and where the limit is taken along any divergent sequence in the manifold.

It is known that (1.1) is insufficient to conclude that the Yamabe problem can be solved. For example, in [5], alongside a number of existence results, Aviles and McOwen give an example of a complete metric g on $\mathbb{R} \times \mathbb{T}^{n-1}$ satisfying (1.1) for which the Yamabe problem has no solution. It is therefore of interest to understand what conditions, in addition to (1.1), are necessary and/or sufficient for the Yamabe problem to have a solution.

Our first result concerns asymptotically locally hyperbolic (ALH) manifolds, a well-studied class of manifolds satisfying (1.1), in a weaker sense than considered in the existing literature (e.g., [1] and [2]). In particular, our notion of ALH requires only C^1 decay in the metric components to those of the model space and a bound on the scalar curvature of the form

$$S_g \leq -n(n-1) + Ce^{-\alpha r} \quad \text{or} \tag{1.2a}$$

$$|S_g + n(n-1)| \leq Ce^{-\alpha r} \tag{1.2b}$$

for $\alpha \in (0, n)$ and some constant $C > 0$, without requiring decay of the full curvature tensor. For the precise definition, see Section 2.1. We establish: