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FINITE VOLUME ELEMENT METHODS: AN OVERVIEW ON RECENT DEVELOPMENTS

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Abstract. In this paper, we present an overview of the progress of the finite volume element (FVE) methods. We show that the linear FVE methods are quite mature due to their close relationship to the linear finite element methods, while development of higher order finite volume methods remains a difficult and promising research front. Theoretical analysis, as well as the algorithms and applications of these methods, are reviewed.

Key words. elliptic equations, finite element methods, finite volume element (FVE) methods, higher order FVE, parabolic problems, Stokes problems

1. Introduction

Finite volume methods have been widely used in sciences and engineering, e.g., computational fluid mechanics and petroleum reservoir simulations. Compared to finite difference (FD) and finite element (FE) methods, finite volume methods are usually easier to implement and offer flexibility in handling complicated domain geometries. More importantly, the methods ensure local mass conservation, a highly desirable property in many applications.

The construction of finite volume methods is based on a balance approach: a local balance is written on each cell which is usually called a control volume; By the divergence theorem, an integral formulation of the fluxes on the boundary of a control volume is obtained; the integral formulation is then discretized with respect to the discrete unknowns.

Finite volume methods have been developed along two directions. First, finite volume methods can be viewed as an extension of finite difference methods on irregular meshes. It is then called cell centered methods or finite difference methods [50]. Such methods usually satisfy the maximum principle and maintain flux consistency. The higher order formulations of cell centered methods need to use a large stencils of neighboring cells in polynomial reconstruction. Second, finite volume methods can be developed in a Petrov-Galerkin form by using two types of meshes: a primal one and its dual, where the primal mesh allows to approximate the exact solution, while the dual mesh allows to discretize the equation. Such finite volume methods are relatively close to finite element methods and are called finite volume element (FVE) methods. FVE methods have the following advantages: 1). the accuracy of FVE methods solely depends on the exact solution and can be obtained arbitrarily by suitably choosing the degree of the approximation polynomials; 2). FVE methods are well suited for complicated domain and require simple treatment to handle boundary conditions. This overview will concentrate on the methodological issues that arise in FVE methods.

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We first use an abstract equation to illustrate the idea of FVE methods. Consider the following equation

$$\mathcal{L}u = f \quad \text{on} \quad \Omega, \tag{1}$$

where $\mathcal{L} : X \longrightarrow Y$ is an operator. Let $\Omega_h = \{K\}$ denote a primal partition of Ω with elements K. Each element is associated with a number of nodes. Nodes are points on the elements K at which linearly independent functionals are prescribed. For each node, we shall associate a domain K^* with it, which is usually called a control volume. All of the control volumes form a dual partition $\Omega_h^* = \{K^*\}$ of Ω. Denote by S_h^* the piecewise test space on Ω_h^*, which is constructed by generalized characteristic functions [70, 71] of control volumes. We recap the definition of generalized characteristic functions here: Let x_0 be a node and D a domain containing x_0. A generalized characteristic functions of D at x_0 comprises the functions which are the polynomial basis functions in the Taylor expansion of a fixed order of a function at x_0 within D , and zero outside D.

Then a variational FVE form for equation (1) is established as

$$(\mathcal{L}u, v_h^*) = (f, v_h^*), \quad \text{for all } v_h^* \in S_h^*. \tag{2}$$

Note that S_h^* contains piecewise constants. Hence, conservation is locally preserved by applying the divergence theorem. The discrete FVE form is to seek an approximation of u in S_h for (2), where S_h is a finite element space defined on Ω_h. Different choices for dual partitions, solution spaces, and test spaces lead to different FVE methods. If piecewise polynomials of degrees k and k' are used for the solution space and the test space, respectively, the corresponding FVE scheme is called $k-k'$ dual grid scheme [23].

2. Linear FVE methods

Linear $(1-0)$ FVE methods have been extensively studied and their theories and algorithms are relatively mature now.

2.1. FVE methods for elliptic problems.

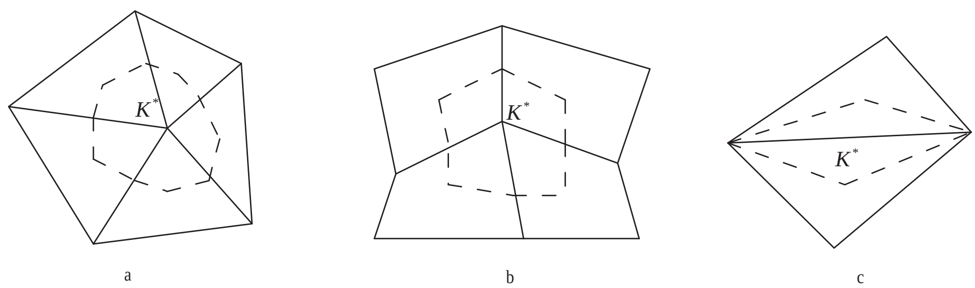

FIGURE 1. A control volume

2.1.1. Conforming, nonconforming, and discontinuous methods. A primal partition for 1D domain (a, b) is denoted by $\Omega_h : a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b$, while its dual is denoted by $\Omega_h^* : a = x_0 < x_{1/2} < \ldots < x_{n-1/2} < x_n = b$ with $(x_{i-1/2}, x_{i+1/2})$ being control volumes. A primal partition Ω_h for 2D linear FVE problems can be constructed using triangles and quadrilaterals. If the solution space S_h is constructed using conforming linear elements, then an associate control volume