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ON GLOBAL ERROR OF SYMPLECTIC SCHEMES FOR STOCHASTIC HAMILTONIAN SYSTEMS

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Abstract. We investigate a first order weak symplectic numerical scheme for stochastic Hamiltonian systems. Given the solution (X_t) and a class of functions f, we derive the expansion of the global approximation error for the computed $E(f(X_t))$ in powers of the discretization step size. The present study is an extension of the results obtained by Talay and Tubaro for the explicit Euler scheme. Based on the derived global error expansion, we construct an extrapolation method of the weak second order. The performance of the extrapolation method is demonstrated numerically for a model simulating oscillations of the particles in storage rings.

Key words. Stochastic Hamiltonian system, symplectic methods, numerical weak schemes, extrapolation method.

1. Introduction

In recent years, considerable progress has been made in the study of uncertainty quantification. It is known that for some practical problems in science and engineering, the effect due to random noise may lead to a significant change in the physical response. Hence, mathematical models based on a deterministic approach may not be sufficient, and the use of stochastic models has been receiving considerable attention. To take into account the random effect, the governing equations are usually represented by stochastic differential equations. In contrast to many efficient and robust numerical algorithms already developed for the deterministic differential equations, the progress on numerical methods for solving stochastic differential equations is less mature. The most common problems associated with computational algorithms for stochastic differential equations are the poor accuracy and poor convergence, especially when long time solutions are required. Hence, it is a challenging task to develop accurate and robust numerical schemes for stochastic differential equations.

The Euler method is a popular numerical method for solving differential equations, and the scheme has been extended to stochastic equations because of its simple implementation [3]. For stochastic Hamiltonian systems, symplectic schemes [5] are important computational methods that preserve the symplectic structure, and their accuracy does not deteriorate even for long time computations. In this paper, we focus on the application of the Euler method and a first order weak symplectic scheme for approximating the solution of stochastic Hamiltonian systems.

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Let consider the autonomous stochastic differential equations (SDEs) in the sense of Stratonovich:

$$
\begin{aligned}
dP &= -\partial_Q H_0(P,Q)dt - \sum_{r=1}^{m} \partial_Q H_r(P,Q) \circ dw_t^r, \quad P(0) = p \\
dQ &= \partial_P H_0(P,Q)dt + \sum_{r=1}^{m} \partial_P H_r(P,Q) \circ dw_t^r, \quad Q(0) = q,
\end{aligned} \tag{1}
$$

where P, Q, p, q are n-dimensional column vectors with the components P_i, Q_i, p_i, q_i, $i = 1, \dots, n$, and $w_t^r, r = 1, \dots, m$ are independent standard Wiener processes. The SDE (1) is called a Stochastic Hamiltonian Systems (SHS) ([7]). Here and in the rest of this paper, for any function f defined on $\mathrm{R}^n \times \mathrm{R}^n$, we denote by $\partial_P f$ the column vector with components $(\partial f/\partial P_i), 1 \le i \le n$, and similarly we let $\partial_Q f$ denote the column vector with components $(\partial f/\partial Q_i), 1 \le i \le n$.

We denote the solution of the stochastic Hamiltonian system (SHS) (1) by $X_t^{0,X_0} = \begin{pmatrix} P_t^{0;p,q} \\ Q_t^{0;p,q} \end{pmatrix}$, where $0 \le t \le T$ and $X_0 = (p^T, q^T)^T$ is a random variable having moments of any order and independent of any increments $W_t - W_s$ of the Wiener process $W_t = (w_t^1, \dots, w_t^r)^T$. It is known that if H_j, $j = 0, \dots, m$ are sufficiently smooth, then X_t^{0,X_0} is a phase flow (diffeomorphism) almost sure ([4]).

The stochastic flow $(p, q) \longrightarrow (P, Q)$ of the SHS (1) preserves the symplectic structure [7, Theorem 2.1] as follows:

$$
dP \wedge dQ = dp \wedge dq, \tag{2}
$$

i.e. the sum of the oriented areas of projections of a two-dimensional surface onto the coordinate planes (p_i, q_i), $i = 1, \dots, n$, is invariant. Here, we consider the differential 2-form

$$
dp \wedge dq = dp_1 \wedge dq_1 + \cdots + dp_n \wedge dq_n,
$$

and differentiation in (1) and (2) have different meanings: in (1) p, q are fixed parameters and differentiation is done with respect to time t, while in (2) differentiation is carried out with respect to the initial data p, q. We say that a method based on the one step approximation $\bar{P} = \bar{P}(t + h; t, p, q)$, $\bar{Q} = \bar{Q}(t + h; t, p, q)$ preserves the symplectic structure if

$$
d\bar{P} \wedge d\bar{Q} = dp \wedge dq.
$$

Let divide the interval $[0, T]$ in N subintervals with a uniform time step $h = T/N$. If the approximation $\bar{X}_0^{0,X_0} = X_0$, $\bar{X}_k^{0,X_0} = (\bar{P}_k^{(0;p,q)}, \bar{Q}_k^{(0;p,q)})$, $k = 1, \dots N$, of the solution $X_{t_k}^{0,X_0} = (P_{t_k}^{(0;p,q)}, Q_{t_k}^{(0;p,q)})$, satisfies

$$
|E[f(\bar{X}_k^{0,X_0})] - E[f(X_{t_k}^{0,X_0})]| \le Kh^j, \tag{3}
$$

for f from a sufficiently large class of functions, where $t_k = kh \in [0, T]$ and the constant $K > 0$ does not depend on k and h, then we say that $\bar{X}_k^{0,X_0}$ approximate the solution $X_{t_k}^{0,X_0}$ of (1) in the weak sense [5] with weak order of accuracy j.

To simplify the notation we define

$$
G_{(r,r)} = \sum_{i=1}^{n} \frac{\partial H_r}{\partial Q_i} \frac{\partial H_r}{\partial P_i}, \tag{4}
$$