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ENERGY STABILITY OF A FIRST ORDER PARTITIONED METHOD FOR SYSTEMS WITH GENERAL COUPLING

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Abstract. We give an energy stability analysis of a first order, 2 step partitioned time discretization of systems of evolution equations. The method requires only uncoupled solutions of sub-systems at every time step without iteration, is long time stable and applies to general system couplings. We give a proof of long time energy stability under a time step restriction relating the time step to the size of the coupling terms.

Key words. partitioned methods, energy stability.

1. Introduction

The most natural approach to numerical simulation of multi-domain, multi-physics systems is to build a partitioned method for the system from components optimized for the individual sub-domain and sub-physics problems. The two most common approaches to partitioning are implicit-explicit methods where the system's' coupling terms are discretized by explicit methods and sub-domain/sub-physics terms by implicit methods, and splitting methods where the coupling terms are themselves separated in each equation according to the subproblems. Application of either to complex problems requires analytic foundations as a guide for practical computation. Herein, we consider the first, implicit-explicit, approach for *general couplings* (the main point herein) but restricted to first order, two step methods. Thus, for a system

$$\begin{aligned} \frac{d}{dt}u_1 + A_1u_1 + B_{11}u_1 + B_{12}u_2 &= f_1, \\ \frac{d}{dt}u_2 + A_2u_2 + B_{21}u_1 + B_{22}u_2 &= f_2, \end{aligned} \tag{1.1}$$

we analyze long-time, energy stability of a method (1.6) below which requires at each step that the two uncoupled linear systems be solved

$$[I + 2\triangle tA_i]u_i^{n+1} = RHS, \;\; i = 1, 2. \tag{1.2}$$

Thus, we consider methods implicit in A_1u_1 and A_2u_2 but explicit in the coupling terms $B_{11}u_1 + B_{12}u_2$ and $B_{21}u_1 + B_{22}u_2$. In the method, the diagonal terms $B_{11}u_1$ and $B_{22}u_2$ could be incorporated into the part treated implicitly (the A_i's). However, the part treated implicitly in (1.3), (1.1) is often determined by existing codes and the coupling terms are those that remain.

Letting $u = (u_1, u_2)^T : [0, \infty) \to \mathbb{R}^d$ and A, B represent the $d \times d$ block matrices in (1.1), we develop the stability analysis for

$$\frac{du}{dt} + Au + Bu = f, u(0) = u_0. \tag{1.3}$$

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Let $\langle \cdot, \cdot \rangle, || \cdot ||$ denote the Euclidean inner product and norm. Suppose

$$A > 0 \text{ i.e., } \langle Au, u \rangle > 0 \text{ for all } u \in \mathbb{R}^d.$$

Partitioned methods, herein, are useful tools and not best for every circumstance. The equally useful alternative is a monolithic method where the fully coupled system is assembled and solved by some iterative method wherein uncoupling can occur in the preconditioning step, e.g., [9] for one example. Conservative couplings ($B^* = -B$, where B^* satisfies $\langle Bx, y \rangle = \langle x, B^* y \rangle \; \forall x, y$) occur when what is lost to one domain or variable is transferred to the other. One important example is the evolutionary Stokes-Darcy model under the Beaver-Joseph-Saffman-Jones (BJSJ) interface condition, [18], [14], [15], [10], [13], [12]. Dissipative couplings ($B = B^* > 0$) occur when there is energy lost through the interaction of the two systems. One important example is in atmosphere-ocean couplings under the rigid lid condition under which there are frictional lossess in transmitting wind energy at the ocean surface to the ocean (and vice versa), e.g., [3], [5], [6], [7]. Another important example of dissipative couplings is compressible flow in a porous medium. The double porosity model of slightly compressible flow in a porous medium [17] is: find $u_1(x,t), u_2(x,t)$

$$\begin{aligned}
(c_1 u_1)_t - \frac{k_1 c_0}{\mu} \triangle u_1 + \alpha^{-1}(u_1 - u_2) &= f, \\
(c_2 u_2)_t - \frac{k_2 c_0}{\mu} \triangle u_2 + \alpha^{-1}(u_2 - u_1) &= g.
\end{aligned}$$

The coupling term satisfies

$$\begin{aligned}
(B\mathbf{u}, \mathbf{u}) &= \int_{\text{flow domain}} \alpha^{-1}(u_1 - u_2)u_1 + \alpha^{-1}(u_2 - u_1)u_2 dx \\
&= \alpha^{-1} \int_{\text{flow domain}} (u_1 - u_2)^2 dx \geq 0
\end{aligned}$$

and is thus dissipative.

One important case where dissipative, conservative and resonant couplings dominated by system dissipation are present is the (above discussed) Stokes-Darcy problem under the orginal (BJ) Beavers-Joesph condition, studied in [4], [16]. Compared to the BJSJ condition, extra terms occur which are resonant and must be sufficiently small in the theory developed in these papers. Building on this previous work, we give combination of these treatments stable for general couplings. The method we study is related to work in [1], [14] and of implicit-explicit type, e.g., [2], [8]. The analytical treatments of component discretizations of the coupling terms in (1.3), (1.1) are known (by a different analytical path for each type of coupling). However, the analysis of energy stability of their combination presents the technical difficulty that one discrete evolution equation with all type present requires one analytical path.

We decompose $B = C + P - N$ (skew symmetric, symmetric positive and symmetric negative parts) and use explicit time discretizations suggested by linear stability theory for each part. Let

$$\text{(1.4)} \qquad \left\{ \begin{array}{c} B = C + P - N \text{ where,} \\ C^* = -C, \quad P^* = P \geq 0, \quad \text{and} \quad N^* = N \geq 0. \end{array} \right.$$