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ENTROPY STABLE SCHEMES FOR COMPRESSIBLE EULER EQUATIONS

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Abstract. A novel numerical flux for the Euler equations which is consistent for kinetic energy and entropy condition was proposed recently [1]. This flux makes use of entropy variable based matrix dissipation which can be shown to satisfy an entropy inequality. For hypersonic flows a blended scheme is proposed which gives carbuncle free solutions for blunt body flows while still giving accurate resolution of boundary layers. Several numerical results on standard test cases using high order accurate reconstruction schemes are presented to show the performance of the new schemes.

Key words. Euler equation, finite volume method, kinetic energy preservation, entropy stability

1. Introduction

The finite volume method for hyperbolic problems requires the specification of a numerical flux function. For scalar problems there is a well developed mathematical theory which provides a route to develop stable and accurate schemes. For systems of conservation laws like Euler equations, the mathematical theory is not complete. Usually the schemes are developed to satisfy certain additional properties like entropy condition and kinetic energy stability which can be important for turbulent flows. Tadmor [17] proposed the idea of entropy conservative numerical fluxes which can then be combined with some dissipation terms using entropy variables to obtain a scheme that respects the entropy condition, i.e., the scheme must produce entropy in accordance with the second law of thermodynamics. However some of these entropy conservative numerical fluxes have to be computed with quadrature rules since the integrals involved in the definition of the flux cannot be evaluated explicitly. For the Euler equations, Roe proposed explicit entropy conservative numerical fluxes [13, 6] which are augmented by Roe-type dissipation terms using entropy variables. These schemes do not suffer from entropy violating solutions that are observed in the original Roe scheme. However for strong shocks, even the first order schemes can produce oscillations indicating that the amount of numerical dissipation is not sufficient. Roe [6] proposed modifying the eigenvalues of the dissipation matrix which lead to non-oscillatory solutions. The modification of the eigenvalues is such that the amount of entropy production is of the correct order of magnitude for weak shocks. The availability of cheap entropy conservative fluxes allows us to use the procedure of [9] to develop high order accurate entropy conservative schemes. Matrix dissipation can be added following the ENO procedure of [2] to develop arbitrarily high order accurate entropy stable schemes for the Euler equations on structured grids.

Faithful representation of kinetic energy evolution is another desirable property of a numerical scheme [8]. This is important for direct numerical simulation (DNS) of turbulent flows where the kinetic energy balance plays an important role in the evolution of turbulence [10, 14, 11]. The scheme is also stable in the sense that

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spurious kinetic energy is not produced by the numerical fluxes. The essential feature for a numerical flux in a semi-discrete finite volume method to correctly capture the kinetic energy balance is that the momentum flux should be of the form $f^m_{j+\frac{1}{2}} = \tilde{p}_{j+\frac{1}{2}} + \overline{u}_{j+\frac{1}{2}} f^\rho_{j+\frac{1}{2}}$ where $\overline{u}_{j+\frac{1}{2}} = (u_j + u_{j+1})/2$ and $\tilde{p}_{j+\frac{1}{2}}, f^\rho_{j+\frac{1}{2}}$ are *any* consistent approximations of the pressure and the mass flux. This scheme thus leaves most terms in the numerical flux unspecified and various authors have used simple averaging. Subbareddy and Candler [16] have proposed a fully discrete finite volume scheme for the compressible Euler equations which preserves kinetic energy but the resulting scheme is implicit. All of these kinetic energy preserving schemes are however not entropy conservative, while on the other hand, the entropy conservative schemes do not have the kinetic energy preservation property. It is thought that for DNS of compressible flows, a numerical scheme which preserves kinetic energy and satisfies entropy condition is desirable since such schemes would be non-linearly stable. Schemes which satisfy entropy condition are found to lead to stable density fluctuations in compressible isotropic turbulence simulations, while schemes which do not have this property can be unstable with respect to density fluctuations [4, 11].

In [1] explicit centered numerical fluxes for the compressible Euler equations which are entropy conservative and also preserve kinetic energy in the case of the semi-discrete finite volume scheme were developed. Due to lack of upwinding, the schemes are not stable for discontinuous solutions and for Navier-Stokes equations on coarse meshes for which shocks may not be well resolved. They yield stable solutions for Navier-Stokes equations when used on very fine meshes where the physical viscosity is enough to stabilize the scheme. However for Euler equations and for Navier-Stokes equations on coarse meshes, the centered fluxes are unstable and must be augmented with dissipation terms. Matrix dissipation similar to Roe scheme but using entropy variables was used to developed entropy stable schemes as in [17]. The eigenvalue modification of Roe [6] is used to compute strong shocks without oscillations. All the schemes are shown to give entropy consistent solutions in cases where the Roe scheme would give entropy violating shocks. The entropy stable schemes with matrix dissipation preserve stationary contacts exactly but also suffer from 1-D shock instability and the carbuncle phenomenon [1]. A modification of the eigenvalues in the dissipation flux based on a blending of the Roe and Rusanov schemes is used which avoids these problems but is still able to accurately compute shear flows like boundary layers. The new schemes are tested here on several standard problems to study their performance. Second order schemes are tested using the MUSCL reconstruction approach and minmod limiter. The blended scheme is shown to give good performance on all the test cases while the basic scheme performs well on problems with weak shocks.

The rest of the paper is organized as follows. Section (2) introduces the 1-D Navier-Stokes equations and finite volume method. This is followed by a discussion of the kinetic energy preservation property and the entropy condition. The new entropy conservative and kinetic energy consistent fluxes are introduced in section (2.5). Matrix dissipation flux is treated in section (3) and modifications to ensure monotone solutions are discussed including a scheme which blends a more accurate scheme with Roe-type eigenvalues, with the Rusanov form of the eigenvalues. Section (4) presents a range of test problems for 1-D shock case involving shocks, expansions and rarefaction solutions. The schemes are compared with other entropy stable schemes and the classical Roe scheme and their performance in the second order version is also demonstrated including under grid refinement.