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# THE NUMERICAL SOLUTION OF DATA ASSIMILATION PROBLEM FOR SHALLOW WATER EQUATIONS

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**Abstract.** The problem on propagation of long waves in a domain of arbitrary form with the sufficiently smooth boundary on a sphere is considered. The boundary consists of "solid" parts passing along the coastline and "liquid" parts passing through the water area. We assume, free surface level observation data on "liquid" boundary is known. In general case the boundary condition on "liquid" part of boundary contains unknown boundary function, which must be found together with component of velocity vector and free surface level. We put an assimilation observation data problem by Prof. V.I. Agoshkov methodology. To solve our ill-posed inverse problem an approach, based on optimal control methods and adjoint equations theory, is used. Numerical solution of direct and adjoint problems is based on finite elements method. Parallel software using MPI is discussed.

**Key words.** data assimilation problem, finite elements method and high performance computation.

## Introduction

Shallow water models adequately describe a large class of natural phenomena such as large-scale free surface waves arising in seas and oceans, tsunamis, flood currents, surface and channel run-offs, gravitation oscillation of the ocean surface [1, 3]. In the papers [3, 4] the numerical modeling of free surface waves in large water areas on the basis of the shallow water equations (SWE) is considered taking into account the Earth's sphericity and the Coriolis acceleration. In [3] for the differential formulation of the problem useful a priori estimates are obtained. This estimates provide stability as well as existence and uniqueness of a solution of the problem. In [4] for this problem the finite elements method (FEM) is constructed and corresponding a priori estimates are obtained. Besides, numerical results on special model grids and on non-structured grids for water areas of the Sea of Okhotsk and the World Ocean are presented. In [5] efficiency of two parallel implementations of the numerical solution of a boundary value problem for SWE with the use of the MPI library for the C language is studied.

In the present paper the problem of long wave propagation in large water area is considered. Mathematical model consists of shallow water equations on spherical surface. Along with the solving the direct problem with special boundary condition at water boundary the inverse problem of rebuilding a function in this boundary condition is considered under given data for deviation of surface. Equations are supplemented by the cost functional including a penalty for deviation from the observed data and regularizing additive. As a result, the process of solving the inverse problem is formulated by an iterative way with the alternate solutions of

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problems with the original and the adjoint operators. The differential problems are reduced to algebraic ones by the finite element method.

Considerable attention is paid to the description of parallel numerical algorithms based on the library MPI and their effectiveness. Performance of two popular implementations of MPI was compared and the behavior of our software is studied when using various ways of memory allocation. In the end of article the convergence of the presented algorithm is illustrated for rebuilding data on the example of the Sea of Okhotsk.

## 1. The differential formulation of a problem

We consider the following problem. Let $(r, \lambda, \varphi)$ be spherical coordinates with the origin at the Earth center. Here $\lambda$ denotes longitude $0 \le \lambda \le 2\pi$ and instead of $\theta$ we use latitude $\varphi = \pi - \theta$, so, $0 \le \varphi < \pi$. We put $r = R_E$ where $R_E$ is the Earth radius which is assumed to be constant.

We formulate the problem on propagation of long waves in a water area as follows. Let $\Omega_{R_E}$ be a given domain on a sphere of radius $R_E$ with the boundary $\Gamma{=}\Gamma_1{\cup}\Gamma_2$ where $\Gamma_1$ is the part of boundary passing along the coast and $\Gamma_2 = \Gamma \setminus \Gamma_1$ is the part of boundary passing through the water area. Denote the characteristic function of these parts of the boundary by $m_1$ and $m_2$, respectively. Without loss of generality we can assume that the points $\varphi = 0$ and $\varphi = \pi$ (poles) are not involved in $\Omega_{R_E}$. Assume also that $\Omega = \{(\lambda, \varphi) \in [0, 2\pi] \times (0, \pi) : (R_E, \lambda, \varphi) \in \Omega_{R_E}\}$. For the unknown functions $u = u(t, \lambda, \varphi)$, $v = v(t, \lambda, \varphi)$ and $\xi = \xi(t, \lambda, \varphi)$ in $\Omega_{R_E} \times (0, T)$ we write the pulse balance equation and the continuity equation [1, 3]:

$$
\begin{aligned}
\frac{\partial u}{\partial t} &= lv + mg\,\frac{\partial \xi}{\partial \lambda} - R_f u + f_1,\\
\frac{\partial v}{\partial t} &= -lu + ng\,\frac{\partial \xi}{\partial \varphi} - R_f v + f_2,\\
\frac{\partial \xi}{\partial t} &= m\left(\frac{\partial}{\partial \lambda}(Hu) + \frac{\partial}{\partial \varphi}\left(\frac{n}{m}Hv\right)\right) + f_3,
\end{aligned}
\tag{1}
$$

where $u$ and $v$ are components of the velocity vector $\mathbf{U}$ in $\lambda$ and $\varphi$ direction, respectively; $\xi$ is deviation of a free surface from the nonperturbed level; $H(\lambda, \varphi) > 0$ is the depth of a water area at a point $(\lambda, \varphi)$; the function $R_f = r_*|\mathbf{U}|/H$ takes into account base friction force, $r_*$ is the friction coefficient; $l = -2\omega\cos\varphi$ is the Coriolis parameter; $m = 1/(R_E \sin\varphi)$; $n = 1/R_E$; $g$ is gravitational acceleration; $f_1 = f_1(t, \lambda, \varphi)$, $f_2 = f_2(t, \lambda, \varphi)$ and $f_3 = f_3(t, \lambda, \varphi)$ are given functions of external forces.

We consider boundary conditions in the following form:

$$
HU_n + \beta m_2\sqrt{gH}\xi = m_2\sqrt{gH}d \quad \text{on} \quad \Gamma \times (0, T),
\tag{2}
$$

where $U_n = \mathbf{U} \cdot \mathbf{n}$, $\mathbf{n} = (n_1, \frac{n}{m}n_2)$ is the vector of an outer normal to the boundary; $\beta \in [0, 1]$ is a given parameter, $d = d(t, \lambda, \varphi)$ is a function defined on the boundary $\Gamma_2$.

We also impose initial conditions

$$
u(0, \lambda, \varphi) = u_0(\lambda, \varphi),\ v(0, \lambda, \varphi) = v_0(\lambda, \varphi),\ \xi(0, \lambda, \varphi) = \xi_0(\lambda, \varphi).
$$

For time discretization we subdivide the segment $[0, T]$ into $K$ subintervals by points:
$0 = t_0 < t_1 < \cdots < t_K = T$ with the step $\tau = T/K$. We approximate time