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MULTISCALE NUMERICAL ALGORITHM FOR 3-D MAXWELL'S EQUATIONS WITH MEMORY EFFECTS IN COMPOSITE MATERIALS

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Abstract. This paper discusses the multiscale method for the time-dependent Maxwell's equations with memory effects in composite materials. The main difficulty is that one cannot use the usual multiscale asymptotic method (cf. [25, 4]) to solve this problem, due to the complication of the memory terms. The key steps addressed in this paper are to transfer the original integro-differential equations to the stationary Maxwell's equations by using the Laplace transform, to employ the multiscale asymptotic method to solve the stationary Maxwell's equations, and then to obtain the computational solution of the original problem by employing a quadrature formula for computing the inverse Laplace transform. Numerical simulations are then carried out to validate the multiscale numerical algorithm in the present paper.

Key words. time-dependent Maxwell's equations, memory effects, multiscale asymptotic expansion, Laplace transform, composite materials.

1. Introduction

The classical macroscopic electromagnetic field is described by four vector-valued functions of position $x \in R^3$ and time $t \in R$ denoted by $\mathbf{E}$, $\mathbf{D}$, $\mathbf{H}$, $\mathbf{B}$. The fundamental field vectors $\mathbf{E}$, $\mathbf{H}$ are the electric and magnetic field intensities, respectively. The vector-valued functions $\mathbf{D}$, $\mathbf{B}$ denote the electric displacement and magnetic induction, respectively. The classical macroscopic Maxwell's equations are given by:

$$\left\{\begin{aligned} &\nabla\times\mathbf{E}+\frac{\partial\mathbf{B}}{\partial t}=0,\\ &\nabla\times\mathbf{H}-\frac{\partial\mathbf{D}}{\partial t}=\mathbf{J},\\ &\nabla\cdot\mathbf{D}=\rho,\\ &\nabla\cdot\mathbf{B}=0,\end{aligned}\right. \tag{1}$$

where $\rho(\mathbf{x},t)$, $\mathbf{J}(\mathbf{x},t)$ are the electric charge density and the source current density, respectively.

The general form of the constitutive laws are the following:

$$\mathbf{D}=\epsilon\mathbf{E}+\int_0^t\Big\{\sigma^{\mathbf{E}}(\mathbf{x})+\nu^{\mathbf{E}}(\mathbf{x},t-\tau)\Big\}\mathbf{E}(\mathbf{x},\tau)d\tau \tag{2}$$

$$\mathbf{B}=\mu\mathbf{H}+\int_0^t\Big\{\sigma^{\mathbf{H}}(\mathbf{x})+\nu^{\mathbf{H}}(\mathbf{x},t-\tau)\Big\}\mathbf{H}(\mathbf{x},\tau)d\tau, \tag{3}$$

where $\epsilon=(\epsilon_{ij})$ and $\mu=(\mu_{ij})$ are the electric permittivity and the magnetic permeability of the media, respectively; $\sigma^{\mathbf{E}}=(\sigma^{\mathbf{E}}_{ij})$, $\nu^{\mathbf{E}}(\mathbf{x},t)$ are the electric conductivity

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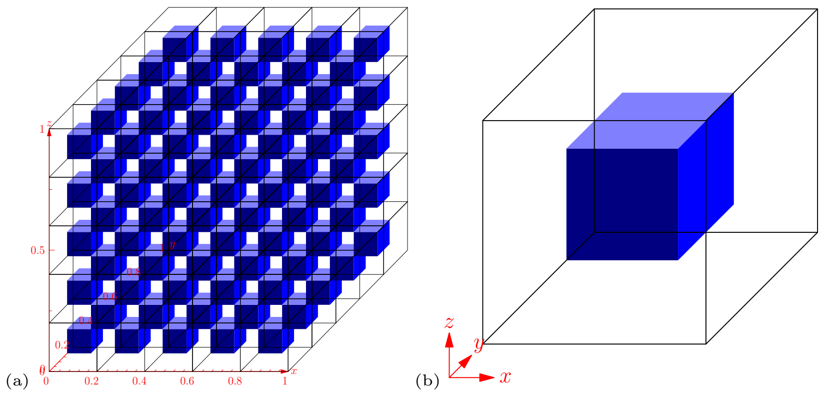

FIGURE 1. (a) Domain Ω; (b) the reference cell Q.

that characterizes the current density and the displacement susceptibility kernel function, respectively;
$\sigma^{\mathbf{H}} = (\sigma^{\mathbf{H}}_{ij})$, $\nu^{\mathbf{H}}(\mathbf{x},t)$ are the magnetic conductivity that characterizes the current density and the magnetic susceptibility, respectively. These are 3×3 positive-definite matrix-valued functions of position $x \in R^3$ in heterogeneous media. In the homogeneous case they are independent of x. In the isotropic case these parameters are scalars or diagonal matrices.

In this paper, we assume that $\sigma^H = \nu^H = 0$. From (1)-(3), by eliminating the magnetic field $\mathbf{H}$, we obtain

$$
\begin{aligned}
&\epsilon(\mathbf{x})\frac{\partial^2 \mathbf{E}(\mathbf{x},t)}{\partial t^2} + (\sigma^{\mathbf{E}}(\mathbf{x}) + \nu^{\mathbf{E}}(\mathbf{x},0))\frac{\partial \mathbf{E}(\mathbf{x},t)}{\partial t} + \nabla\times(\mu^{-1}(\mathbf{x})\nabla\times\mathbf{E}) \\
&+\frac{\partial \nu^{\mathbf{E}}}{\partial t}(\mathbf{x},0)\mathbf{E}(\mathbf{x},t) + \int_0^t \frac{\partial^2 \nu^{\mathbf{E}}(\mathbf{x},t-\tau)}{\partial t^2}\mathbf{E}(\mathbf{x},\tau)d\tau = -\frac{\partial}{\partial t}J(\mathbf{x},t),
\end{aligned} \tag{4}
$$

where $\mu^{-1}(\mathbf{x})$ denotes the inverse matrix of $\mu(\mathbf{x})$.

Suppose that $\Omega \subset R^3$ is a bounded polygonal convex domain or a smooth domain with a Lipschitz continuous boundary $\partial\Omega$ with a periodic microstructure as illustrated in Fig.1 (a) and (b). For convenience, we replace $\nabla\times\mathbf{u}$ with $\mathbf{curl}\,\mathbf{u}$. We then consider the following Maxwell's equations with rapidly oscillating coefficients:

$$
\begin{cases}
B(\frac{\mathbf{x}}{\varepsilon})\frac{\partial^2 \mathbf{E}^\varepsilon(\mathbf{x},t)}{\partial t^2} + C(\frac{\mathbf{x}}{\varepsilon})\frac{\partial \mathbf{E}^\varepsilon(\mathbf{x},t)}{\partial t} + G(\frac{\mathbf{x}}{\varepsilon})\mathbf{E}^\varepsilon(\mathbf{x},t) + \mathbf{curl}(A(\frac{\mathbf{x}}{\varepsilon})\mathbf{curl}\,\mathbf{E}^\varepsilon) \\
\quad + \int_0^t K(\frac{\mathbf{x}}{\varepsilon}, t-\tau)\mathbf{E}^\varepsilon(\mathbf{x},\tau)d\tau = \mathbf{f}(\mathbf{x},t), \quad (\mathbf{x},t)\in\Omega\times(0,T) \\
\mathbf{E}^\varepsilon \times \mathbf{n} = 0, \quad (\mathbf{x},t)\in\partial\Omega\times(0,T) \\
\mathbf{E}^\varepsilon(\mathbf{x},0) = \mathbf{E}_0(\mathbf{x}), \quad \frac{\partial \mathbf{E}^\varepsilon(\mathbf{x},0)}{\partial t} = \mathbf{E}_1(\mathbf{x}),
\end{cases} \tag{5}
$$

Here ε denotes a small periodic parameter, which is the relative size of the unit cell. The matrix-valued functions $A(\frac{\mathbf{x}}{\varepsilon})$, $B(\frac{\mathbf{x}}{\varepsilon})$, $C(\frac{\mathbf{x}}{\varepsilon})$, $G(\frac{\mathbf{x}}{\varepsilon})$, $K(\frac{\mathbf{x}}{\varepsilon}, t-\tau)$, and the vector-valued functions $\mathbf{f}(\mathbf{x},t)$, $\mathbf{E}_0(\mathbf{x})$, $\mathbf{E}_1(\mathbf{x})$ are known functions, $\mathbf{n} = (n_1,n_2,n_3)$ is the outward unit normal to $\partial\Omega$.

We first define the **curl** of a distribution $\mathbf{u} = (u_1,u_2,u_3)$ of $\mathcal{D}'(\Omega)^3$ by

$$
\mathbf{curl}\ \mathbf{u} = (\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}, \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}, \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}).
$$