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DATA DEPENDENT STABILITY OF FORWARD IN TIME AND CENTRED IN SPACE (FTCS) SCHEME FOR SCALAR HYPERBOLIC EQUATIONS

RITESH KUMAR DUBEY

Abstract. The main novelty of this note is the approach which is used to show that Forward Time and Centred in Space (FTCS) scheme is data dependent stable for scalar hyperbolic conservation laws. Note that FTCS is well known to be unconditionally unstable in von-Neumann sense. In this new approach, the ratio of consecutive gradients is used to classify the initial data region where FTCS is non-oscillatory and stable. Numerical results for 1D scalar and system test problems are given to verify the claim.

Key words. Numerical oscillations; von-Neumann stability; smoothness parameter; finite difference schemes; hyperbolic conservation laws.

1. Introduction

The notion of stability of a numerical scheme for the time dependent problems evolves around the induced spurious numerical oscillations especially at discontinuities. In the founding work, Courant-Friedrichs and Levy shown that for the convergence, a difference scheme must contains the physical domain of dependence of the partial differential equation [1]. In other words, they gave a necessary condition on the ratio of the spatial and time discritization steps for the stability of the difference scheme known as CFL condition. Later O'Brien, Hyman and Kaplan defined the stability of a difference scheme in terms of the growth of rounding errors [14]. In the seminal work [13], Lax and Richtmyer defined the stability using the uniformly boundedness of linear difference operator in the numerical scheme and gave the the necessary and sufficient condition for the convergence of linear schemes. One can summarize that the stability of a scheme ensures for the bounded growth of the solution. This becomes a significant requirement when it comes to approximate the solution of following scalar hyperbolic initial value problem

$$u(x,t)_t + g(u(x,t))_x = 0,\ u(x,0) = u_0(x). \tag{1}$$

This boundedness of numerical solution is required because the physical solution $u(x,t)$ of (1) satisfies the following maximum principle,

$$\min_x(u(x,0)) \le u(x,t) \le \max_x u(x,0),\ \forall x \text{ and } t \ge 0. \tag{2}$$

If one consider a uniform grid with the spatial width h, time step k and denote the discrete mesh point (x_j,t_n) by $x_j = jh,\ j = 1,2,\dots N$ and $t_n = nk,\ k = 1,2,\dots M$. Then the CFL number for (1) can be defined as $C = \lambda \max_u |g'(u)|$ where $\lambda = \frac{k}{h}$ and $g'(u)$ is the characteristic speed associated with (1). It can be approximated

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at the cell interface $x_{j+\frac{1}{2}}$ of the cell $[x_j, x_{j+1}]$ can by

$$a_{j+\frac{1}{2}} = \begin{cases} \frac{\Delta_+ g_j^n}{\Delta_+ u_j^n} & \text{if } \Delta_+ u_j^n \neq 0, \\ g_j' & else. \end{cases} \tag{3}$$

Starting from the the work in [1], the CFL number has been an indispensable tool for defining the stability of numerical schemes. The linear von-Neumann stability analysis of a numerical scheme deduces a stability condition on the CFL number. Stronger non-linear stability conditions which ensure the boundedness requirement (2) in the numerical solution of (1) are also heavily depend on the CFL number. For example upwind range condition, monotone stability [12, 2, 20], positivity preserving [8, 16] and total variation stability [6].

In fact, one needs to satisfy the CFL requirement i.e., $C \leq K$ where $K \in \mathbb{R}$ to devise any new stable scheme e.g., TVD schemes in [3, 9, 19], essentially non-oscillatory (ENO) schemes [7, 21], weighted essentially non-oscillatory schemes [24]. Apart from defining the stability, in a recent paper [10], the CFL number is exploited for the improved approximation by the flux limiters based scheme.

In this work, we consider the FTCS scheme obtained by the discretization of (1) by replacing the time derivative with a forward difference and the space derivative with a centred difference formula as

$$u_j^{n+1} = u_j^n - \frac{k}{2h}\left(g_{j+1}^n - g_{j-1}^n\right). \tag{4}$$

where $g_j^n = g(u_j^n)$ and $u_j^n \approx u(x_j, t_n)$. The above three point centred FTCS scheme (4) seems to be a correct and natural choice as the spatial discritization in FTCS does not violets the physical domain of dependence of (1). Contrary to the expectation, even for the linear problem $g(u) = au$, the solution obtained by FTCS scheme (4) is diverging and the induced oscillations grow exponentially no matter how small the time step is compared to the space step. The classical von-Neumann stability analysis also shows that FTCS (4) is unconditionally unstable. Moreover, FTCS does not satisfy criteria for any of the above mentioned non-linear stability conditions see [11]. One can find such unconditional unstability of FTCS scheme (4) quite surprising mainly because as

- The FTCS (4) and the centered Lax-Wendroff (LxW) scheme [15] shares the same spatial stencil of grid points. Note that, for the CFL number $C = \lambda|a| \leq 1$, the three point centred LxW scheme is linearly stable [23] while FTCS is completely unstable.
- It can be observed that for smooth initial data, such as sinusoidal wave, the induced oscillations by FTCS does not grow immediately. Moreover, up to some extent, the occurrence of induced oscillations can be controlled by choosing small CFL number C, see Figure 2. On the other hand when applied on discontinuous initial data, FTCS introduces strong oscillations immediately see Figure 3(a).

These observations have been the motivation for the present study of the dependence of induced oscillations by the FTCS scheme (4) on data type and the CFL number. More precisely, we look for initial data type for which FTCS preserves the positivity, monotonicity and is local extremum diminishing stability properties. In order to carry out the analysis, we follow the idea used by the author in [4]. Note that the schemes analyzed in [4] are **stable** in the von-Neumann sense where as the FTCS scheme (4) considered in this work is **unconditionally unstable**.