Analysis in Theory and Applications
Anal. Theory Appl., Vol. **29**, No. 3 (2013), pp. 267-274
DOI: 10.4208/ata.2013.v29.n3.6

Endpoint Estimates for Hardy Operator's Conjugate Operator with Power Weight on n-Dimensional Space

Xudong Nie[1], Shimo Wang[2] and Dunyan Yan[1,*]

[1] *School of Mathematics Science, University of Chinese Academy of Sciences, Zhongguancun, Beijing 100190, China*
[2] *School of Mathematics Science, Heilongjiang University, Harbin 150080, Heilongjiang, China*

Received 29 January 2013; Accepted (in revised version) 2 September 2013

Available online 30 September 2013

Abstract. In this paper, we establish two integral inequalities for Hardy operator's conjugate operator at the endpoint on n-dimensional space. The operator H_n^* is bounded from $L^1_{x^\alpha}(\mathbb{G}^n)$ to $L^q_{x^\beta}(\mathbb{G}^n)$ with the bound explicitly worked out and the similar result holds for $\mathcal{H}_n^*$.

Key Words: Conjugate operator, power weight, endpoint estimate.

AMS Subject Classifications: 42B20, 42B35

1 Introduction

Let f be a non-negative integrable function on $\mathbb{G}:=(0,\infty)$. The classical Hardy operator is defined by

$$Hf(x):=\frac{1}{x}\int_0^x f(t)dt,$$

and its conjugate operator

$$H^*f(x):=\int_x^\infty \frac{f(t)}{t}dt$$

for all $x>0$.

For n-dimensional case with $n\geq 2$, Hardy operators can be defined on product space as

$$H_nf(x):=\frac{1}{x_1\cdots x_n}\int_0^{x_1}\cdots\int_0^{x_n} f(t_1,\cdots,t_n)dt_1\cdots dt_n, \tag{1.1}$$

*Corresponding author. *Email addresses:* `niexudong10@mails.ucas.ac.cn` (X. Nie), `wangshimo2012@sina.cn` (S. Wang), `ydunyan@ucas.ac.cn` (D. Yan)

and its conjugate operator defined as

$$H_n^* f(x) := \int_{x_1}^{\infty} \cdots \int_{x_n}^{\infty} \frac{f(t_1,\cdots,t_n)}{t_1\cdots t_n} dt_1 \cdots dt_n, \tag{1.2}$$

for $x=(x_1,x_2,\cdots,x_n)\in\mathbb{G}^n=(0,\infty)^n$, where f is any measurable function on $\mathbb{G}^n$.

Another definition is given by Christ and Grafakos in [2] as follows

$$\mathcal{H}_n f(x) = \frac{1}{\omega_n |x|^n} \int_{|y|<|x|} f(y)dy, \tag{1.3}$$

and

$$\mathcal{H}_n^* f(x) = \frac{1}{\omega_n} \int_{|y|>|x|} \frac{f(y)}{|y|^n} dy, \tag{1.4}$$

for $x\in\mathbb{R}^n\backslash\{0\}$, where f is any measurable function on $\mathbb{R}^n$ and $\omega_n=\pi^{n/2}/\Gamma(1+n/2)$ is the volume of the unit ball in $\mathbb{R}^n$.

For the case $1<p\le q<\infty$, the boundedness of the operators H_n and H_n^* from $L^p_{x^\alpha}(\mathbb{G}^n)$ to $L^q_{x^\beta}(\mathbb{G}^n)$ were discussed in many papers (cf. [1,13,15,17]). The estimates on the endpoint for the operator H_n and $\mathcal{H}_n$ were systematically studied in [18]. It should be pointed out that the operator H_n, $n\ge 2$, defined by (1.1) fails to be of weak type of $(1,1)$, however, H_n is bounded from $L^1_{x^\alpha}(\mathbb{G}^n)$ to $L^q_{x^\beta}(\mathbb{G}^n)$ for arbitrary $n\in\mathbb{N}$. This shows that some power weight can change the boundedness of the operator H_n on the endpoint. Motivated by the idea of the reference [18], it is natural for us to discuss the the boundedness of the operator H_n^* on the endpoint.

The purpose of this paper is to establish the boundedness of the operators H_n^* and $\mathcal{H}_n^*$ on the endpoint.

Throughout the paper, we have the following notations. For two n-dimensional vectors $\alpha=(\alpha_1,\alpha_2,\cdots,\alpha_n)$ and $\beta=(\beta_1,\beta_2,\cdots,\beta_n)$, $\alpha\cdot\beta=\alpha_1\beta_1+\alpha_2\beta_2+\cdots+\alpha_n\beta_n$, $\alpha<\beta$ means each $\alpha_i<\beta_i$, $i=1,\cdots,n$, and $x^\alpha=x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n}$, $x\in\mathbb{G}^n$. For some bold type figures and letters, we have $\mathbf{1}=(1,\cdots,1)$, $\mathbf{p}=(p,\cdots,p)$. It is clear that $x^{\mathbf{1}}=x_1\cdots x_n$.

Now we first formulate our main results as follows.

Theorem 1.1. *Suppose that f is any non-negative measurable function on $\mathbb{G}^n$ and $1\le q<\infty$. If α and β are two n-tuples in $\mathbb{R}^n$ such that $\alpha+\mathbf{1}>\mathbf{0}$ and $\beta+\mathbf{1}=q(\alpha+\mathbf{1})$, then the following inequality*

$$\left(\int_{\mathbb{G}^n} (H_n^* f(x))^q x^\beta dx\right)^{\frac{1}{q}} \le \left(\prod_{i=1}^n \frac{1}{(\beta_i+1)}\right)^{\frac{1}{q}} \int_{\mathbb{G}^n} f(x)x^\alpha dx \tag{1.5}$$

holds for the operator H_n^* defined by (1.2)*, that is, H_n^* is bounded from $L^1_{x^\alpha}(\mathbb{G}^n)$ to $L^q_{x^\beta}(\mathbb{G}^n)$ with the norm of H_n^* satisfying*

$$\|H_n^*\| \le \left(\prod_{i=1}^n \frac{1}{(\beta_i+1)}\right)^{\frac{1}{q}}.$$