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Semi-stability for Holomorphic Bundles with Global Sections

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Abstract. In this paper, we establish a generalized Hitchin-Kobayashi correspondence between the τ-semi-stability and the existence of approximate τ-Hermitian-Einstein structure on holomorphic pair (E,ϕ). As an application, we obtain the Bogomolov type inequality for τ-semi-stable holomorphic pair.

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Key Words: τ-semi-stable; approximate τ-Hermitian-Einstein structure; holomorphic pair.

1 Introduction

The Hitchin-Kobayashi correspondence states that a holomorphic vector bundle over a compact Kähler manifold is stable if and only if it is simple and admits an Hermitian-Einstein metric. This correspondence was first proved for compact Riemannian surfaces by Narasimhan-Seshadri [1], for algebraic manifolds by Donaldson [2], and for general Kähler manifolds by Uhlenbeck-Yau [3]. The classical Hitchin-Kobayashi correspondence has several interesting and important generalizations and extensions where some extra structures are added to the holomorphic bundles, for example: [4–8], etc.

In [9], Kobayashi introduced the notion of approximate Hermitian-Einstein structure on a holomorphic vector bundle, and he proved that a holomorphic vector bundle with an approximate Hermitian-Einstein structure must be semi-stable. Furthermore, over projective algebraic manifolds, Kobayashi solved the inverse problem, i.e. the semi-stability implies admitting an approximate Hermitian-Einstein structure. The general Kähler case was studied by Jacob ([10]). Recently, Li and Zhang ([6]) showed that the

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semi-stability of Higgs bundles also implies the existence of approximate Hermitian-Einstein structure by using the heat flow method.

Let (M,ω) be a compact Kähler manifold. Given a coherent sheaf $\mathcal{F}$ on M, the ω-slope $\mu(\mathcal{F})$ of $\mathcal{F}$ is defined by

$$\mu(\mathcal{F})=\frac{\deg_\omega(\mathcal{F})}{\mathrm{rank}\mathcal{F}}=\frac{1}{\mathrm{rank}\mathcal{F}}\int_M c_1(\det\mathcal{F})\wedge\frac{\omega^{n-1}}{(n-1)!}. \tag{1.1}$$

where $\det\mathcal{F}$ is the determinant line bundle of $\mathcal{F}$. A holomorphic pair (E,ϕ) over M is a holomorphic bundle E coupled with ϕ a holomorphic global section of E. For a real number τ, a holomorphic pair (E,ϕ) is called τ-stable (τ-semi-stable) if it satisfies the following two conditions:

(i) It holds $\mu(\mathcal{E}')<(\leq)\frac{\tau\mathrm{Vol}(M)}{4\pi}$ for all reflexive sub-sheaves $\mathcal{E}'\subset E$.

(ii) $\frac{r\mu(E)-r'\mu(\mathcal{E}')}{r-r'}>(\geq)\frac{\tau\mathrm{Vol}(M)}{4\pi}$, for every reflexive sub-sheaf $\mathcal{E}'$ with $0<\mathrm{rank}(\mathcal{E}')=r'<\mathrm{rank}(E)=r$ such that $\phi\in\mathcal{E}'$ almost everywhere.

Bradlow ([11]) first investigated the following vortex equation

$$\sqrt{-1}\Lambda_\omega F_H+\frac{1}{2}\phi\otimes\phi^{*H}-\frac{1}{2}\cdot\tau\mathrm{Id}=0, \tag{1.2}$$

where H is an Hermitian metric on E, F_H is the curvature of the Chern connection with respect to H, Λ_ω denotes the contraction of differential forms by Kähler form ω, ϕ^{*H} is the adjoint of ϕ with respect to H and τ is a real parameter. An Hermitian metric satisfying (1.2) will be called a τ-Hermitian-Einstein metric. In [11], Bradlow established a correspondence between the τ-stability of holomorphic pair (E,ϕ) and the existence of τ-Hermitian-Einstein metric.

We say a holomorphic pair (E,ϕ) admits an approximate τ-Hermitian-Einstein structure if for every positive ϵ, there is an Hermitian metric H such that

$$\max_M\left|\sqrt{-1}\Lambda_\omega F_H+\frac{1}{2}\phi\otimes\phi^{*H}-\frac{1}{2}\tau\mathrm{Id}\right|_H<\epsilon. \tag{1.3}$$

In this paper, we establish a correspondence between the τ-semi-stability and the existence of approximate τ-Hermitian-Einstein structure on holomorphic pair (E,ϕ), i.e. we prove the following theorem.

Theorem 1.1. *Assume (E,ϕ) is a holomorphic pair on compact Kähler manifold (M,ω), then (E,ϕ) is τ-semi-stable if and only if it admits an approximate τ-Hermitian-Einstein structure.*

Let $c_1(E)$ and $ch_2(E)$ be the first Chern class and second Chern character of E respectively, and set

$$C_1(E)=\int_M c_1(E)\wedge\frac{\omega^{n-1}}{(n-1)!},\quad Ch_2(E)=\int_M ch_2(E)\wedge\frac{\omega^{n-2}}{(n-2)!}. \tag{1.4}$$