INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 11, Number 1, Pages 34–53

ON THE TIME APPROXIMATION OF THE STOKES EQUATIONS WITH NONLINEAR SLIP BOUNDARY CONDITIONS

J. K. DJOKO

Abstract. This work is concerned with the numerical approximation of the unsteady Stokes flow of a viscous incompressible fluid driven by a threshold slip boundary condition of friction type. The continuous problem is formulated as variational inequality, which is next discretize in time based on backward Euler's scheme. We prove existence and uniqueness of the solution of the time discrete problem by means of a regularization approach. Finally, we derive error estimates that justify the convergence property of the discretization proposed.

Key words. eywords: Stokes equations, slip boundary condition, variational inequality, regularization, convexity, monotone

1. Introduction

We consider unsteady flows of incompressible viscous fluids modeled by the S-tokes system

$$\boldsymbol{u}_t - 2\nu \operatorname{div} \varepsilon(\boldsymbol{u}) + \nabla p = \boldsymbol{f} \quad \text{in } Q = \Omega \times (0, T), \tag{1.1}$$

$$\operatorname{div} \boldsymbol{u} = 0 \quad \text{in } Q, \tag{1.2}$$

where Ω is the flow region, a bounded domain in $\mathbb{R}^2$, while $\varepsilon(\boldsymbol{u}) = \frac{1}{2}[\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T]$. The motion of the incompressible fluid is described by the velocity $\boldsymbol{u}(\boldsymbol{x}, t)$ and pressure $p(\boldsymbol{x}, t)$. In (1.1) $\boldsymbol{f}(\boldsymbol{x}, t)$ is the external body force per unit volume, while ν is the kinematic viscosity. Equations (1.1) and (1.2) are supplemented by boundary and initial conditions. We first assume that

$$\boldsymbol{u}(\boldsymbol{x}, 0) = \boldsymbol{u}_0 \quad \text{on } \overline{\Omega}, \tag{1.3}$$

where $\boldsymbol{u}_0$ is a given function, for which precise assumptions will be introduced below, and $\overline{\Omega}$ is the closure of Ω. Next in order to describe the motion of the fluid at the boundary, we assume that the boundary of Ω, say, $\partial\Omega$ is made of two components S (say the outer wall) and Γ (the inner wall), and we require that $\overline{\partial\Omega} = \overline{S \cup \Gamma}$, with $S \cap \Gamma = \emptyset$. We assume the homogeneous Dirichlet condition on Γ, that is

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma \times (0, T). \tag{1.4}$$

We have chosen to work with homogeneous condition on the velocity in order to avoid the technical arguments linked to the Hopf lemma (see [1], Chapter 4, Lemma 2.3). On S, we first assume the impermeability condition

$$u_N = \boldsymbol{u} \cdot \boldsymbol{n} = 0 \text{ on } S \times (0, T), \tag{1.5}$$

Received by the editors January 1, 2012 and, in revised form, March 7, 2013.
2000 *Mathematics Subject Classification.* 65M12, 76D07, 35J85, 35Q30, 76D30, 76D07.
The author thank the referees for pertinent remarks that have led to some improvements of this study. I also thank our colleague Dr Christiaan Leroux for some stimulating discussions at the beginning of this project.

where $\boldsymbol{n}$ is the outward unit normal on the boundary $\partial\Omega$, and u_N is the normal component of the velocity,while $\boldsymbol{u_\tau} = \boldsymbol{u} - u_N\boldsymbol{n}$ is its tangential component. In addition to (1.5) we also impose on S, a threshold slip condition [2, 3, 4], which is the main ingredient of this work. The threshold slip condition can be formulated with the knowledge of a positive function $g : S \longrightarrow (0,\infty)$ which is called the barrier of threshold function and the tangential part of $\boldsymbol{Tn}$ as follows:

(1.6)
$$\left.\begin{array}{llll} \text{if} & |(\boldsymbol{Tn})_{\boldsymbol{\tau}}| < g & \text{then} & \boldsymbol{u_\tau} = \boldsymbol{0}, \\ \text{if} & |(\boldsymbol{Tn})_{\boldsymbol{\tau}}| = g & \text{then} & \boldsymbol{u_\tau} \neq \boldsymbol{0}\ ,\ \textit{and}\ -(\boldsymbol{Tn})_{\boldsymbol{\tau}} = g\dfrac{\boldsymbol{u_\tau}}{|\boldsymbol{u_\tau}|} \end{array}\right\} \text{ on } \quad S\times(0,T).$$

Of course in (1.6), $\boldsymbol{T} = 2\nu\varepsilon(\boldsymbol{u}) - p\boldsymbol{I}$ is the Cauchy stress tensor with $\boldsymbol{I}$ being the identity tensor. It should quickly be mentioned that (1.6) is equivalent to [5]

$$(\boldsymbol{Tn})_{\boldsymbol{\tau}}\cdot\boldsymbol{u_\tau} + g|\boldsymbol{u_\tau}| = 0 \text{ on } \quad S\times(0,T), \tag{1.7}$$

which is re-written with the use of sub-differential as

$$-(\boldsymbol{Tn})_{\boldsymbol{\tau}} \in\ g\partial|\boldsymbol{u_\tau}| \quad \text{on } S\times(0,T), \tag{1.8}$$

where $\partial|\cdot|$ is the sub-differential of the real valued function $|\cdot|$, with $|\boldsymbol{w}|^2 = \boldsymbol{w}\cdot\boldsymbol{w}$. We recall that if $\boldsymbol{X}$ is a Hilbert space equipped with the inner product denoted as $\cdot$, and $x_0 \in \boldsymbol{X}$, then

$$y \in \partial\Psi(x_0) \text{ if and only if } \Psi(x) - \Psi(x_0) \geq y\cdot(x - x_0) \quad \forall x \in \boldsymbol{X}. \tag{1.9}$$

The slip boundary conditions of friction type (1.6) can be justified by the fact that frictional effects of the fluid at the pores of the solid can be very important, and this can be seen in fiber spinning. Hence one observes that different boundary conditions describe different physical phenomena. The boundary condition (1.6) has also been applied successfully to some flow phenomena in concrete situations such as oil flow over or beneath sand layers [6, 7]. In [8], a generalization of the boundary condition (1.6) is formulated and analyzed for the steady Stokes flow, while the case of Navier-Stokes equations has been examined in [9]. We should re-iterate that for fluids with moderate velocities and stresses, the no-slip condition is well suited and describe the fact that the fluid adheres to the boundary of the flow domain.

The subject of the present work is to approximate the two-dimensional problem (1.1)$\cdots$(1.6) in time, using the implicit Euler scheme, and establish its well-posedness, stability and measure the difference between the exact and discrete solution. We also want our scheme to have some properties observed at the continuous level. The existence theory of (1.1)$\cdots$(1.6) provided in [2, 3, 4] used semi-group approach, so no estimates of the solution are available in that research. Hence for completeness, we revisit the existence and uniqueness question by adopting the Galerkin's approach together with the energy method. By doing so, we have some a priori estimates that we would like our numerical scheme to have. Similar studies have been presented for the case of Navier-Stokes equations with Dirichlet or periodic boundary conditions in [10, 11, 12, 13, 14], reaction diffusion equations and parabolic p-Laplacian in [15, 16]. To better understand the analysis of the time discrete problem, it is important to present the main steps of the existence result of (1.1)$\cdots$ (1.6) which is done by regularization approach, Faedo-Galerkin approximation and using some compactness arguments [5]. The regularization is important because we have a non differentiable term which brings the inequality into the system. But also from the numerical analysis viewpoint, the regularization itself is worth considering (as we are going to see, the solution of the regularized and