INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 11, Number 1, Pages 148–171

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Computing and Information

CONVERGENCE OF A RELAXATION SCHEME FOR A 2×2 TRIANGULAR SYSTEM OF CONSERVATION LAWS

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Abstract. We study relaxation approximations to solutions of a 2×2 triangular system of conservation laws. We show that smooth relaxation approximations exist for all time. A finite difference approximation of the relaxation system gives rise to a relaxation scheme of the Jin and Xin type. In both cases we show that a sequence of approximate solutions is produced where the limit is a weak solution of the triangular system. Compensated compactness is used to establish convergence.

Key words. triangular systems of conservation laws, relaxation, compensated compactness

1. Introduction

The aim of this paper is to prove convergence of two sequences of functions approximating a weak solution (u, v) of the 2×2 hyperbolic system

$$(1)\quad \begin{cases} u_t + f(u)_x = 0, \\ v_t + g(u,v)_x = 0, \end{cases} \qquad x \in \mathbb{R},\, t > 0, \qquad (u,v)(x,0) = (u_0(x), v_0(x)),\ \ x \in \mathbb{R},$$

where the flux functions f and g, and the initial data are known. This type of system arises in models of three-phase flow in porous media, see [9]. Systems of this kind are called *triangular*, since the first equation is independent of the second. An interesting class occurs if $f \equiv 0$, so that u acts as a coefficient which may be discontinuous. In recent years, conservation laws with discontinuous coefficients has received considerable attention, see e.g. [1, 10] and the references therein. For scalar conservation laws with a discontinuous coefficient, i.e., $f \equiv 0$ in (1), [1] outlines a theory of well-posedness. We emphasise that no such theory exists if $f \not\equiv 0$, and while a corollary of the convergence proved in this paper is the existence of weak solutions of (1), our methods yield no information regarding uniqueness or continuous dependence on the initial data for this weak solution.

The first sequence $\{(u^\varepsilon, v^\varepsilon)\}$ for which we prove convergence consists of the solutions of the weakly coupled strictly hyperbolic relaxation system:

$$(2)\qquad \begin{cases} u^\varepsilon_t + w^\varepsilon_x = 0, \\ w^\varepsilon_t + a^2 u^\varepsilon_x = \frac{1}{\varepsilon}\left(f(u^\varepsilon) - w^\varepsilon\right), \\ v^\varepsilon_t + z^\varepsilon_x = 0, \\ z^\varepsilon_t + b^2 v^\varepsilon_x = \frac{1}{\varepsilon}\left(g(u^\varepsilon, v^\varepsilon) - z^\varepsilon\right), \end{cases} \qquad x \in \mathbb{R},\, t > 0,$$

with initial condition

$$(3)\qquad (u^\varepsilon, w^\varepsilon, v^\varepsilon, z^\varepsilon)(x,0) = (u^\varepsilon_0, w^\varepsilon_0, v^\varepsilon_0, z^\varepsilon_0)(x) = (u^\varepsilon_0, f(u^\varepsilon_0), v^\varepsilon_0, g(u^\varepsilon_0, v^\varepsilon_0))(x),$$

for $x \in \mathbb{R}$. We consider the triangular system (1) as an equilibrium for the Cauchy problem (2) - (3) as first introduced by Liu in [13]. The second sequence for which we prove convergence is made by applying a finite difference scheme to the relaxation system (2). The main advantage in construction a numerical scheme in this manner

Received by the editors June 27, 2012, and, in revised form, May 2, 2013 .
2000 *Mathematics Subject Classification.* 35L45, 35L40, 65M06.

is that one does not rely on solving local Riemann problems when approximating solutions of (1). Moreover, the scheme is explicit leaving it easy to implement.

The relaxation approximation $(u^\varepsilon, w^\varepsilon)$ given by (2) has been shown to converge strongly to $(u, f(u))$ where u is the entropy solution of the single conservation law

$$u_t + f(u)_x = 0,\ u(x,0) = u_0(x),\ x \in \mathbb{R},\ t > 0. \tag{4}$$

See for instance [15, 16]. In [7] a numerical scheme was constructed based on the relaxation approximations, and convergence to the entropy solution of (4) was proved in [2]. In this paper we extend these results to hold for the relaxation approximations and a relaxation scheme for the complete 2×2 system (1). In particular, we show that a subsequence $\{(v^{\varepsilon_n}, z^{\varepsilon_n})\}_{n\in\mathbb{N}}$ of solutions of (2) converges in L^p_{loc}, $1 \le p < \infty$, to a weak solution of

$$v_t + g(u,v)_x = 0,\ v(x,0) = v_0(x), \tag{5}$$

where u is the entropy solution of (4), and similarly in the numerical case. In [9] finite volume schemes was used to construct approximate solutions of (1), and convergence of a subsequence to a weak solution was shown following the compensated compactness approach of [11] where convergence of the Lax-Friedrichs scheme was established for conservation laws with a discontinuous space-time dependent flux. Finite volume schemes has also been used to approximate solutions to the relaxation approximations (2) for a general $n \times n$ system, see [3].

If $f \equiv 0$ and $u(x) = u_0(x)$ is some BV function, our numerical scheme will reduce to the relaxation scheme in [8]. In both works [8, 9] convergence to a weak solution of (1) of some sequences of approximate solutions given by the respective schemes, was proved under some strong CFL conditions depending on the flux functions. In order to prove convergence of the relaxation scheme we will also need to introduce such a strengthened CFL condition. The fact that the approximations of the entropy solution of (4) is of bounded variation in space uniformly in the approximation parameters is crucial when proving convergence. A major difficulty in extending the results to hold for relaxation approximations to solutions of $n \times n$ triangular systems, is that we have no BV estimates on the approximations of the function v in (1). This lack of regularity also seems an obstacle in obtaining existence results for such triangular systems.

We will consider the system (1) under a set of assumptions, presented in the following section, which are needed to assure that solutions are bounded. Moreover, the main stability criterion is motivated by a Chapman-Enskog expansion, which in the case of system (2) reads within an $\mathcal{O}(\varepsilon^2)$ term

$$\begin{aligned} u^\varepsilon_t + f(u^\varepsilon)_x &= \varepsilon\left[\left(a^2 - f'(u^\varepsilon)^2\right) u^\varepsilon_x\right]_x, \\ v^\varepsilon_t + g(u^\varepsilon, v^\varepsilon)_x &= \varepsilon\left[\left(b^2 - g_v(u^\varepsilon, v^\varepsilon)^2\right) v^\varepsilon_x\right]_x \\ &\quad - \varepsilon\left[\left(f'(u^\varepsilon) g_u(u^\varepsilon, v^\varepsilon) + g_u(u^\varepsilon, v^\varepsilon) g_v(u^\varepsilon, v^\varepsilon)\right) u^\varepsilon_x\right]_x. \end{aligned} \tag{6}$$

Equation (6) gives us a first order correction to (1). For the equations to be parabolic we need $\left(a^2 - f'(u^\varepsilon)\right) \ge 0$ and $\left(b^2 - g_v(u^\varepsilon, u^\varepsilon)\right) \ge 0$. The condition

$$|f'(u)| < a \text{ and } |\partial_u g(u,v)|,\ |\partial_v g(u,v)| < b,$$

is called the subcharacteristic condition and is due to Whitham [17], Liu [13] and Chen, Levermore and Liu [5]. Note also that the variables w^ε and z^ε in (2) can be eliminated. The result is a system of two conservation laws that has been