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Remarks on the Regularity Criteria of Three-Dimensional Navier-Stokes Equations in Margin Case

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Abstract. In the study of the regularity criteria for Leray weak solutions to three-dimensional Navier-Stokes equations, two sufficient conditions such that the horizontal velocity $\tilde{u}$ satisfies $\tilde{u}\in L^2(0,T;BMO(\mathbf{R}^3))$ or $\tilde{u}\in L^{2/1+r}(0,T;\dot{B}^r_{\infty,\infty}(\mathbf{R}^3))$ for $0<r<1$ are considered.

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1 Introduction and main results

The incompressible fluid motion in the whole space $\mathbf{R}^3$ is governed by the Navier-Stokes equations with unit viscosity

$$\begin{cases} \partial_t u+(u\cdot\nabla)u+\nabla\pi=\Delta u, \\ \nabla\cdot u=0, \\ u(x,0)=u_0. \end{cases} \tag{1.1}$$

Here $u=(u_1,u_2,u_3)$ and π present the unknown velocity field and the unknown scalar pressure field, u_0 is a given initial velocity.

Since the pioneer study of Leray [1] in 1930s, there is a large literature on the well-posedness of weak solutions to the incompressible Navier-Stokes equations. Many contributions have been made in an effort to understand the regularity of the weak solutions. However, the problem on the regularity or finite time singularity for the weak solution

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still remains unsolved. Regularity can only been derived when certain growth conditions are satisfied. This is known as a regularity criterion problem. The investigation of the regularity criterion on the weak solution stems from the celebrated work of Serrin [2]. With the extended examinations given by Struwe [3], Serrin's regularity criterion can be described as follows:

A weak solution u is regular if the growth condition

$$u\in L^p(0,T;L^q(\boldsymbol{R}^3))\equiv L^pL^q, \qquad \textit{for} \quad \frac{2}{p}+\frac{3}{q}=1, \quad 3<q\le\infty, \tag{1.2}$$

holds true.

The condition described by (1.2) which involves all components of the velocity vector field $u=(u_1,u_2,u_3)$ is known as degree -1 growth condition (see Chen and Xin [4]), since

$$\|u(\lambda\cdot,\lambda^2\cdot)\|_{L^pL^q}=\|u\|_{L^p(0,\lambda^2T;L^q(\mathbf{R}^3))}\lambda^{-\frac{2}{p}-\frac{3}{q}}=\|u\|_{L^p(0,\lambda^2T;L^q(\mathbf{R}^3))}\lambda^{-1}.$$

The degree -1 growth condition is critical due to the scaling invariance property. That is, $u(x,t)$ solves (1.1) if and only if $u_\lambda(x,t)=\lambda u(\lambda x,\lambda^2t)$ is a solution of (1.1).

Moreover, this result has been extended by many authors in terms of velocity $u(x,t)$, the gradient of velocity $\nabla u(x,t)$ or vorticity $w(x,t)=\nabla\times u$ in Lebesgue spaces, BMO space or Besov spaces, respectively (refer to [5–10] and reference therein).

Actually, the weak solution remains regular when a part of the velocity components is involved in some growth conditions. For example, regularity of the weak solution was recently obtained by Beirão da Veiga [11] (see also Dong and Chen [12]) when the horizontal velocity denoted by

$$\tilde{u}=(u_1,u_2,0)$$

satisfies the critical growth condition

$$\tilde{u}\in L^pL^q, \qquad \text{for} \quad \frac{2}{p}+\frac{3}{q}=1, \quad 3<q\le\infty. \tag{1.3}$$

And some critical growth conditions on the two vorticity components were obtained by Kozono and Yatsu [13], Zhang and Chen [14]. One may also mentioned that the weak solution remains regular if the single velocity component satisfies the higher (subcritical) growth conditions (see Zhou [15, 16], Penel and Pokorý [17], Kukavica and Ziane [18], Cao and Titi [19]).

The margin case $q=\infty$ in (1.3) appears to be more challenging. The aim of the present paper is to improve the regularity criterion (1.3) from Lebesgue space L^∞ to BMO space and Besov space (see the definitions in Section 2), respectively.

Before statement the main results, we firstly recall the definition of Leray weak solution of Navier-Stokes equations (see, for example, [20]).

Definition 1.1. *Let $u_0\in L^2(\boldsymbol{R}^3)$ and $\nabla\cdot u_0=0$. A vector field $u(x,t)$ is termed as a Leray weak solution of* (1.1) *if u satisfies the following properties:*