JOURNAL OF PARTIAL DIFFERENTIAL EQUATIONS
J. Part. Diff. Eq., Vol. **22**, No. 3, pp. 205-217

doi: 10.4208/jpde.v22.n3.2
August 2009

Source Type Solutions of a Fourth Order Degenerate Parabolic Equation

LIU Changchun, LIAN Songzhe* and WANG Zhao

Department of Mathematics, Jilin University, Changchun 130012, China.

Received 16 September 2008; Accepted 14 July 2009

Abstract. In this paper, we study a generalized thin film equation which is relevant to capillary driven flows of thin films of power-law fluids. We prove that the generalized thin film equation in dimension $d \geq 2$ has a unique C^1 source type radial self-similar nonnegative solution if $0 < n < 2p-1$ and has no solution of this type if $n \geq 2p-1$.

AMS Subject Classifications: 35K30, 35K55, 76A20

Chinese Library Classifications: O175.26

Key Words: Source type solutions; degenerate parabolic equation; existence.

1 Introduction

In this paper, we consider the nonnegative source type solutions of the generalized thin film equation

$$\frac{\partial h}{\partial t} + \operatorname{div}(h^n |\nabla \Delta h|^{p-2} \nabla \Delta h) = 0, \quad p > 2, \quad \text{in } \mathbb{R}^d \times (0,\infty), \tag{1.1}$$

namely, the nonnegative solutions of (1.1) satisfying

$$h(\cdot,t) \to M\delta \quad \text{as } t \to 0^+, \tag{1.2}$$

where $d \geq 2$, Δ is the Laplacian, δ stands for the *Dirac* mass and n, p and M are positive constants.

By definition, (1.2) means that

$$\int_{\mathbb{R}^d} h(x,t)\varphi(x)\mathrm{d}x \to M\varphi(0), \quad \text{as } t \to 0^+ \text{ for all } \varphi \in C_0(\mathbb{R}^d), \tag{1.3}$$

*Corresponding author. *Email addresses:* `liucc@jlu.edu.cn` (C. Liu), `liansz@jlu.edu.cn` (S. Lian)

and we shall require, in addition, the point-wise convergence

$$h(x,t)\to 0, \quad \text{as } t\to 0^+ \text{ for all } x\neq 0. \tag{1.4}$$

Eq. (1.1) is a typical higher order equation which has a strong physical background and a rich theoretical connotation. It is relevant to capillary driven flows of thin films of power-law fluids, where $h(x,t)$ denotes the height from the surface of the oil to the surface of the solid [1]. King [1] studied the Cauchy problem of the equation in one-dimension, exploited local analysis about the edge of the support and special closed form solutions such as traveling waves, separable solutions and instantaneous source solutions, see also [2]. Ansini and Giacomelli [3] studied the free-boundary problem of Eq. (1.1) in one-dimensional case, and obtained the existence of solutions on a multi-step approximation procedure. However, the higher dimensional case remains open problem.

When $d=2$ and $p>2$, Liu, Yin and Gao [4] investigated the existence, uniqueness and asymptotic behavior of generalized solutions for Eq. (1.1) with $n=0$ to initial-boundary problem.

During the past years, much attention has been paid to study the source type solutions [5–7]. However, only a few papers devoted to the source type solutions of the higher order equation. Bernis et al. [8] studied the source type solutions of Eq. (1.1) with $p=2$ in one-dimension; see also Beretta [9] for a related equation. Ferreira and Bernis [10], Bernis and Ferreira [11] consider the source type solutions of Eq. (1.1) with $p=2$, for $d\geq 2$.

We look for solutions of the form

$$h(x,t)=t^{-d\beta}f(r), \quad r=|x|t^{-\beta}, \quad \beta=\frac{1}{(4+d)(p-1)+d(n-1)}. \tag{1.5}$$

Then the function $f=f(r)$ is a solution of the problem

$$\Big(r^{d-1}\left[f^n|(\Delta_r f)'|^{p-2}(\Delta_r f)'\right]\Big)'=\beta\Big(r^d f\Big)', \quad r>0, \tag{1.6}$$

$$r^d f(r)\to 0, \quad \text{as } r\to\infty, \tag{1.7}$$

$$\omega_d\int_0^\infty r^{d-1}f(r)\mathrm{d}r=M, \tag{1.8}$$

where

$$\Delta_r=\frac{\mathrm{d}^2}{\mathrm{d}r^2}+\frac{d-1}{r}\frac{\mathrm{d}}{\mathrm{d}r}$$

is the radial Laplacian and ω_d is the area of the unit sphere in $\mathbb{R}^d$.

To make precise the meaning of this problem we introduce the conditions

(H1) $f(r)$ is C^1 for all $r>0$ and C^3 if $f(r)>0$ and $r>0$;
(H2) $f^n(\Delta_r f)'$ has a C^1 extension to $(0,\infty)$;
(H3) $f(r)$ is C^1 for $r=0$ and $f'(0)=0$.

We will prove the following results.