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NECESSARY AND SUFFICIENT CONDITIONS FOR OSCILLATIONS OF NEUTRAL HYPERBOLIC PARTIAL DIFFERENTIAL EQUATIONS WITH DELAYS*

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Abstract This paper is concerned with the oscillations of neutral hyperbolic partial differential equations with delays. Necessary and sufficient conditions are obtained for the oscillations of all solutions of the equations, and these results are illustrated by some examples.
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1. Introduction

The oscillatory properties of partial differential equations with delays are applied widely in many fields, such as in biology, engineering, medicine, physics, chemistry and etc. In recent years, the discussion about these properties has become a hot topic. At the same time, many sufficient conditions for oscillations are obtained. We mention here [1,2,3]. However, only a few necessary and sufficient conditions for oscillations of such equations are established, in particular, about the hyperbolic differential equations. In this paper, our aim is to study the necessary and sufficient conditions for oscillations of all solutions about the hyperbolic partial differential equation

$$\begin{aligned}&\frac{\partial^2}{\partial t^2}[u(t,x)+\sum_{i=1}^{m}p_i(t)u(t-\sigma_i(t),x)]+\frac{\partial}{\partial t}[u(t,x)+\sum_{j=1}^{n}q_j(t)u(t-\tau_j(t),x)]\\&\quad=a_0(t)\Delta u(t,x)+\sum_{k=1}^{l}a_k(t)\Delta u(t-\rho_k(t),x)-\sum_{s=1}^{h}r_s(t)u(t-\mu_s(t),x),\\&\qquad(t,x)\in R_+\times\Omega \qquad (1)\end{aligned}$$

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with the boundary conditions as follows

$$u(t,x)=0,(t,x)\in R_+\times\partial\Omega, \tag{2}$$

$$\frac{\partial u}{\partial n}=0,(t,x)\in R_+\times\partial\Omega, \tag{3}$$

where Ω is a bounded domain in R^n with a piecewise smooth boundary $\partial\Omega$, $G\equiv R_+\times\Omega$.

Throughout this paper we assume the conditions (H) are satisfied: $p_i(t),q_j(t)\in C(R_+,R)$; $a_0(t),a_k(t),r_s(t),\sigma_i(t),\tau_j(t),\rho_k(t),\mu_s(t)\in C(R_+,(0,\infty))$; $0<\sigma_i(t)\le\sigma, 0<\tau_j(t)\le\tau, 0<\rho_k(t)\le\rho, 0<\mu_s(t)\le\mu$, where $\sigma,\tau,\rho,\mu=const.$,$i\in\{1,2,\cdots,m\}=I_m, j\in I_n, k\in I_l, s\in I_h$.

A solution $u(t,x)$ of the problem (1),(2)(or (1),(3)) means that $u(t,x)\in C^2(G)\cap C^1(\overline{G})$ and satisfies equation (1) in the domain G and the boundary (2)(or (3)); the solution $u(t,x)$ of the problem (1),(2)(or (1),(3)) is called to be oscillatory in the domain G if for any number $T>0$, there exists a point $(t_0,x_0)\in[T,\infty)\times\Omega$ such that $u(t_0,x_0)=0$ holds.

2. Main Results

Now we investigate the necessary and sufficient conditions for oscillations of problems (1),(2) and (1),(3). The following lemmas are useful in the proof of our main results.

Lemma 1 *Let b be a constant, b_0 be the minimum eigenvalue of the problem*

$$\begin{cases}\Delta\varphi(x)+b\varphi(x)=0, x\in\Omega;\\ \varphi(x)=0, x\in\partial\Omega,\end{cases}$$

and $\varphi(x)$ be its corresponding characteristic function. Then $b_0>0$ and $\varphi(x)>0$ for $x\in\Omega$.

Lemma 2 *The necessary and sufficient condition for oscillations of all solutions of the neutral differential equation (4) is that its characteristic equation $F(\lambda)=0$ has no real roots.*

$$[v(t)+\sum_{i=1}^{m}p_iv(t-\tau_i)]''+[v(t)+\sum_{j=1}^{n}r_jv(t-\mu_j)]'+\sum_{k=1}^{d}q_kv(t-\sigma_k)=0, \tag{4}$$

$$F(\lambda)=\lambda^2(1+\sum_{i=1}^{m}p_ie^{-\lambda\tau_i})+\lambda(1+\sum_{j=1}^{n}r_je^{-\lambda\mu_j})+\sum_{k=1}^{d}q_ke^{-\lambda\sigma_k}=0,$$

where $p_i,r_j\in R,\tau_i,q_k>0,\mu_j,\sigma_k\ge 0$ are all constants,$i\in I_m, j\in I_n, k\in I_d$.

The proof of Lemma 2 is similar to references [4,5]. We omit it here.