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FINITE VOLUME MULTILEVEL APPROXIMATION OF THE SHALLOW WATER EQUATIONS WITH A TIME EXPLICIT SCHEME

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Abstract. We consider a simple advection equation in space dimension one and the linearized shallow water equations in space dimension two and describe and implement two different multilevel finite volume discretizations in the context of the utilization of the incremental methods with time explicit or semi-explicit schemes.

Key words. finite volume methods, multilevel methods, Euler explicit schemes, shallow water equations, stability analysis.

1. Introduction

This article is related to the article [5] in which we investigated multilevel finite volume discretizations for the one dimensional advection equation and for the one and two-dimensional linear shallow water equations. This article is also related to the article [2] in which we presented and implemented a hierarchical multilevel finite volume discretization for the shallow water equations combined with a Runge-Kutta discretization of order four in time. The article [5] focused on the Euler implicit time discretization, this article continues with the stability analysis of the multilevel finite volume methods but with a partly or fully Euler explicit discretization in time.

We consider the simple one-dimensional advection equation and the full two-dimensional shallow water equations without viscosity, linearized around a constant flow. For the shallow water equations the boundary conditions and the analysis depend on the nature of the background flow; see [13] and below. In this article we choose the supercritical case which allows us to use a classical upwind finite volume scheme, see e.g. [14].

Our motivations are two-fold. On the physical side the shallow water equations are a simplified model of the Primitive Equations (PEs) of the atmosphere and the oceans. As shown in [21], [18], in a rectangular geometry, the PEs can be expanded using a certain vertical modal decomposition; with such a decomposition we obtain an infinite system of coupled equations which resemble the shallow water equations. See e.g. [8], [9] for the actual numerical resolution of these coupled systems. However it appears in these articles that the problems to be solved are very difficult (demanding) and performant numerical methods are needed to tackle more and more realistic problems. We turned in [2] to multilevel finite volume methods which are here our second motivation. Finite volume methods are desirable for the treatment of complicated geometrical domains such as the oceans, and multilevel methods of the incremental unknown type are useful for the implementation of multilevel methods. Such methods have been introduced in the context of the Nonlinear Galerkin Method in [15] (see also [16]), in the context of finite

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differences in [20], and in the context of spectral methods and turbulence in [11]. In continuation of [2], this article explores the finite volume implementation of the incremental unknowns.

Considering, for simplicity, a rectangular geometry, we divide our domain in "small" cells of size Δx for the one dimensional case and of size $\Delta x \times \Delta y$ for the two dimensional case which we combine at the first level of increment, in coarse cells of size $3\Delta x$ and $3\Delta x \times 3\Delta y$ respectively. The unknowns on the small cells are the original unknowns denoted by u or $\mathbf{u}$, and we also introduce, for the coarse cells, suitable averaged values of the unknowns denoted by U or $\mathbf{U}$. We also introduce the incremental unknowns, denoted by Z or $\mathbf{Z}$, which are frozen during the computation on the coarse mesh and which allow us to go from the unknowns on the coarse mesh to the unknowns on the fine mesh.

We apply different time steps on the fine mesh and on the coarse mesh. Since the cells are smaller on the fine mesh we use a smaller time step, $\Delta t/p$, where p is chosen, and we use a time step Δt for our computation on the coarse mesh. This coarsening can be repeated once more considering cells of size $9\Delta x$ or $9\Delta x \times 9\Delta y$, and possibly several times as the programming is repetitive and its cost is thus small; however as done in [5] we restrict ourselves in this article to one coarsening.

The stability analysis developed here is done on a multilevel method that is different than that presented in [2] and closer to that presented in [5] (see however below and in Section 5). At the end of this article we numerically compare the method presented in this article with the averaged multilevel method used in [2] and [5].

Of course there is a very rich literature on the discretization of the shallow water equations using multilevel and/or parallel methods; see e.g. [1], [3], [10], [12], [17], [22], and the references therein.

This article is organized as follows. In Section 2 we present the hierarchical multilevel discretization for the one dimensional advection equation. For the time discretization we use the Euler explicit or semi-explicit method. Then in Section 3 we investigate a hierarchical multilevel discretization for the two dimensional linear shallow water equations. In Section 4 we re-introduce the Averaged Multilevel Finite Volume method presented in [2] and [5] for the advection equation. We discuss several questions related to the stability of the method that we also investigate numerically. Finally, we present some numerical results on the two dimensional linear shallow water equations comparing computations done solely on the fine grid, computations only done on the coarse mesh and computations done with the hierarchical multilevel method and the averaged multilevel method.

2. Hierarchical Multilevel Finite Volume Method I

We present in this section a hierarchical multilevel method using a finite volume discretization (HFVM) for the following advection equation on the one-dimensional domain $\mathcal{M} = (0, L)$:

$$\frac{\partial u}{\partial t}(x,t) + \frac{\partial u}{\partial x}(x,t) = 0, \;\; x \in \mathcal{M}, \;\; t > 0. \tag{1}$$

This equation is supplemented with the boundary condition

$$u(0,t) = 0, \;\; t > 0, \tag{2}$$

and the initial condition

$$u(x,0) = u^0(x), \;\; x \in \mathcal{M}, \tag{3}$$