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FINITE VOLUME APPROXIMATION OF THE LINEARIZED SHALLOW WATER EQUATIONS IN HYPERBOLIC MODE

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Abstract. In this article, we consider the linearized inviscid shallow water equations in space dimension two in a rectangular domain. We implement a finite volume discretization and prove the numerical stability and convergence of the scheme for three cases that depend on the background flow $\tilde{u}_0$, $\tilde{v}_0$, and $\tilde{\phi}_0$ (sub- or super-critical flow at each part of the boundary). The three cases that we consider are fully hyperbolic modes.

Key words. shallow water equations, finite volume method, stability, and convergence.

1. Introduction

This article aims to study the finite volume approximation of the initial and boundary value problem for the linearized shallow water (SW) equations in a rectangle. This article builds on two previous articles [15] and [9]. In the theoretical paper [15] the authors determine all the boundary conditions that one can associate to the linearized shallow water equations and find, as explained below, five different situations depending on the respective values of $\tilde{u}_0$, $\tilde{v}_0$, $\tilde{\phi}_0$ corresponding to the (constant) background flow around which the linearization is made. Omitting the non generic cases where one of these constants vanish, we can assume, by a change of variables that $\tilde{u}_0$, $\tilde{v}_0$, $\tilde{\phi}_0$ are > 0. The article [15] raises of course the question of the approximation of the SW equations in the rectangle in these different situations. This question was investigated in [9] which considers the approximation of the inviscid linearized shallow water equations in the so-called supercritical (supersonic) case, that is when $\tilde{u}_0^2 + \tilde{v}_0^2 > g\tilde{\phi}_0$ (see below). Four cases remain to be studied and we consider in this article three of them for which the stationary part of the SW equations are fully hyperbolic. We do not discuss in this article the approximation of the fifth case for which the stationary part of the SW equations is partly hyperbolic and partly elliptic as this case necessitates a different approach.

Theoretically, we extended the results in [15] to more general hyperbolic systems in [16] and possibly to more general polygonal-like domains in the fully hyperbolic case (see [16, Remark 2.3]). Hence, we could also study the finite volume approximation in the more general setting. However, in this article, we prefer to consider the shallow water equations in a rectangular domain to stay close from our initial motivation of this work that is the study of the Local Area Models (LAMs) in the atmosphere and oceans sciences, see e.g. [22].

The linearized shallow water equations that we consider read

$$(1.1)\qquad \begin{cases} u_t + \tilde{u}_0 u_x + \tilde{v}_0 u_y + g\phi_x = f_u, \\ v_t + \tilde{u}_0 v_x + \tilde{v}_0 v_y + g\phi_y = f_v, \\ \phi_t + \tilde{u}_0 \phi_x + \tilde{v}_0 \phi_y + \tilde{\phi}_0 (u_x + v_y) = f_\phi, \end{cases}$$

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where $(x, y) \in \mathcal{M} := (0, L_x) \times (0, L_y)$, (u, v) are the horizontal components of the velocity and ϕ is the potential height. The advection velocities $\tilde{u}_0$, $\tilde{v}_0$ and the mean geopotential height $\tilde{\phi}_0$ are constants, g is the gravitational acceleration, f_u, f_v, and f_ϕ are the source terms. As shown in [15], the boundary conditions which can be associated with these equations depend on the relative values of the velocities ($\tilde{u}_0^2, \tilde{v}_0^2 >$ or $< g\tilde{\phi}_0$), that is whether these velocities are sub- or supercritical (sub- or supersonic). The three supersonic cases, when $\tilde{u}_0^2 + \tilde{v}_0^2 - g\tilde{\phi}_0 > 0$, that we consider are called: the mixed hyperbolic case (two sub-cases) and the fully hyperbolic subcritical case. The supercritical case, when $\tilde{u}_0 > \sqrt{g\tilde{\phi}_0}$, $\tilde{v}_0 > \sqrt{g\tilde{\phi}_0}$, has been considered in [9]. In this article we will focus on the other three cases. For the mixed hyperbolic case, we only consider one sub-case, where

$$\tilde{u}_0,\, \tilde{v}_0,\, \tilde{\phi}_0 > 0, \quad \tilde{u}_0 < \sqrt{g\tilde{\phi}_0},\;\; \tilde{v}_0 > \sqrt{g\tilde{\phi}_0}, \tag{1.2}$$

since the other sub-case where

$$\tilde{u}_0,\, \tilde{v}_0,\, \tilde{\phi}_0 > 0, \quad \tilde{u}_0 > \sqrt{g\tilde{\phi}_0},\;\; \tilde{v}_0 < \sqrt{g\tilde{\phi}_0},$$

would be similar. In the fully hyperbolic subcritical case, we assume that

$$\tilde{u}_0,\, \tilde{v}_0,\, \tilde{\phi}_0 > 0, \quad \tilde{u}_0 < \sqrt{g\tilde{\phi}_0},\;\; \tilde{v}_0 < \sqrt{g\tilde{\phi}_0},\;\; \tilde{u}_0^2 + \tilde{v}_0^2 - g\tilde{\phi}_0 > 0. \tag{1.3}$$

We will study the cases (1.2) and (1.3) separately in Section 2 and 3.

As we know, the literature on the shallow water equations is very vast, both on the theoretical and computational aspects, considering the viscous equations or the partly or totally inviscid equations and considering that the height is either always strictly positive or that it can vanish. See e.g. [1,2,4,8,12,21] on the computational side and see e.g. [5,6,10,11,14,17–20] on the theoretical side. Regarding the numerical stability of time discretized finite volume schemes, see e.g. [8], [9], and [12]. The proof of the convergence results follows the same methods as e.g. [3] and [13].

This article is organized as follows. At the end of this introductory section, we present some notations which we will use throughout this article. Section 2 and 3 are devoted to show the stability and convergence results of the finite volume scheme for the linearized SW equations in the mixed hyperbolic case and in the fully hyperbolic subcritical case, respectively.

We now write (1.1) in the compact form

$$\mathbf{u}_t + \mathcal{E}_1 \mathbf{u}_x + \mathcal{E}_2 \mathbf{u}_y = \mathbf{f}, \tag{1.4}$$

where $\mathbf{u} = (u, v, \phi)^T$, $\mathbf{f} = (f_u, f_v, f_\phi)^T$ and

$$\mathcal{E}_1 = \begin{pmatrix} \tilde{u}_0 & 0 & g \\ 0 & \tilde{u}_0 & 0 \\ \tilde{\phi}_0 & 0 & \tilde{u}_0 \end{pmatrix}, \quad \mathcal{E}_2 = \begin{pmatrix} \tilde{v}_0 & 0 & 0 \\ 0 & \tilde{v}_0 & g \\ 0 & \tilde{\phi}_0 & \tilde{v}_0 \end{pmatrix}.$$

Note that $\mathcal{E}_1$, $\mathcal{E}_2$ admit a symmetrizer $S_0 = \mathrm{diag}(1, 1, g/\tilde{\phi}_0)$, which means that $S_0\mathcal{E}_1, S_0\mathcal{E}_2$ are both symmetric (see e.g. [7, Chapter 1]).

Here and in the following, we endow the space $H = L^2(\mathcal{M})^3$ with the Hilbert scalar products and norms, for $\mathbf{u} = (u, v, \phi)^T$, $\mathbf{u}' = (u', v', \phi')^T$:

$$\begin{aligned} &\langle \mathbf{u}, \mathbf{u}' \rangle = (S_0 \mathbf{u}, \mathbf{u}') = (u, u') + (v, v') + \frac{g}{\tilde{\phi}_0}(\phi, \phi'), \quad && |\mathbf{u}| = \{\langle \mathbf{u}, \mathbf{u} \rangle\}^{1/2}, \\ &(\mathbf{u}, \mathbf{u}') = \mathbf{u}'^T \mathbf{u} = (u, u') + (v, v') + (\phi, \phi'), && \|\mathbf{u}\| = \{(\mathbf{u}, \mathbf{u})\}^{1/2}. \end{aligned} \tag{1.5}$$